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Pitch and Efield Systematics

Pitch

Our unperturbed system is a muon precessing in a uniform magnetic field with zero electric field. We use a cylindrical coordinate system where $\mathbf{B} = B\hat{\mathbf{z}}$. The rate of change of the polarization with respect to the muon velocity is described by Jackson 11.171

$$\frac{d}{dt}(\hat{\boldsymbol{\beta}} \cdot \mathbf{s}) = -\frac{e}{mc}\mathbf{s}_\perp \cdot \left[a_\mu \hat{\boldsymbol{\beta}} \times \mathbf{B} + \left(a_\mu - \frac{m^2c^2}{p^2}\right)\beta\mathbf{E}\right] \tag{1}$$

Our trial solution for time dependence of the polarization in the ρ, ϕ plane is $\mathbf{s} = |\mathbf{s}|(\cos\omega_a t\hat{\boldsymbol{\rho}} - \sin\omega_a t\hat{\boldsymbol{\phi}})$ where in the unperturbed system $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\phi}}$. We find that

$$\mathbf{s}_\perp = \mathbf{s} - (\hat{\boldsymbol{\beta}} \cdot \mathbf{s}) \cdot \hat{\boldsymbol{\beta}} = |\mathbf{s}| \cos\omega_a t\hat{\boldsymbol{\rho}}$$

and

$$\hat{\boldsymbol{\beta}} \cdot \mathbf{s} = -|\mathbf{s}| \sin\omega_a t$$

Substitution into 1 with the above assumptions gives us

$$\omega_a = \frac{e}{mc}Ba_\mu$$

So far so good. We note that the component of the polarization vector in the $\rho - \phi$ plane rotates about the z-axis. The component perpendicular to that plane is fixed. And if there happens to be a component of velocity that is parallel to the magnetic field, the polarization vector will continue rotating about the z-axis, confined to the $\rho - \phi$ plane.

Now we consider the effect of perturbing the system so that the z-component of the velocity, $\hat{\boldsymbol{\beta}}$ is oscillating in the electrostatic field of the quadrupoles. That is

$$\hat{\boldsymbol{\beta}} = \cos\psi(t)\hat{\boldsymbol{\phi}} + \sin\psi(t)\hat{\mathbf{z}}$$

where $\psi(t)$ is the angle of the trajectory with respect to the $\rho - \phi$ plane. If we suppose for a moment that there is no magnetic field, the muon will execute betatron oscillations in the focusing field of the quadrupoles. The orientation of the polarization of the magic momentum muon with respect to the velocity is frozen. Namely,

$$\frac{d}{dt}(\hat{\boldsymbol{\beta}} \cdot \mathbf{s}) = 0.$$

Evidently

$$\mathbf{s} = |\mathbf{s}| \left([\cos\psi(t)\hat{\boldsymbol{\phi}} + \sin\psi(t)\hat{\mathbf{z}}] \sin\alpha + \hat{\boldsymbol{\rho}}\cos\alpha\right) \tag{2}$$

where α is some arbitrary time independent angle. Then as required

$$\frac{d}{dt}\left(\hat{\boldsymbol{\beta}} \cdot \mathbf{s}\right) = \frac{d}{dt}\left(|\mathbf{s}| \sin\alpha\right) = 0$$

The spin vector is frozen and follows the velocity vector identically.

Next restore the uniform and time independent magnetic field and assume $\psi(t) = \psi_0 \cos\omega_p t$. Let's imagine that at the peak of the betatron oscillation, where the velocity is perpendicular to the magnetic field, ($\hat{\boldsymbol{\beta}}\cdot\mathbf{B}=0$), that the polarization $\mathbf{s}$ is in the $\rho-\phi$ plane. If the E-field dominates ($\omega_p \gg \omega_a$) then as the pitch angle increases, the polarization will evolve a z-component that follows the z-component of velocity. If the magnetic field dominates ($\omega_a \gg \omega_p$), the polarization will be fixed in the $\rho-\phi$ plane. If $\omega_p \gg \omega_a$, the z-component of polarization will follow the z-component of velocity. More generally, in the $\omega_p \gg \omega_a$ limit, the effect of the magnetic field is that the z-component of the polarization lags (or leads) the z-component of the velocity. If $\omega_p \gg \omega_a$ the plane of the polarization vector will oscillate about the $\hat{\boldsymbol{\rho}}$ direction. We generalize the expression for the spin vector (Equation 2) to

$$\mathbf{s} = |\mathbf{s}|\left([\cos(\psi(t)-\Delta\psi(t))\hat{\boldsymbol{\phi}} + \sin(\psi(t)-\Delta\psi(t))\hat{\mathbf{z}}]\sin\alpha + \hat{\boldsymbol{\rho}}\cos\alpha\right).$$

Then

$$\mathbf{s}\cdot\hat{\boldsymbol{\beta}} = |\mathbf{s}|(\cos(\psi+\Delta\psi)\cos\psi + \sin(\psi+\Delta\psi)\sin\psi)\sin\alpha = |\mathbf{s}|\cos\Delta\psi\sin\alpha,$$

where $\Delta\psi(t)$ represents the phase offset. Substitution into Equation 1 gives

$$\begin{aligned} |\mathbf{s}|\frac{d}{dt}(\sin\alpha\cos(\Delta\psi)) &= -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\alpha\cos\psi \quad (3)\\ |\mathbf{s}|(\dot{\alpha}\cos\alpha\cos(\Delta\psi) - \sin\alpha(\dot{\Delta\psi})\sin\psi) &= -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\alpha(\cos\psi) \quad (4)\end{aligned}$$

where $\dot{\alpha}, \dot{\Delta\psi}$ indicate derivative with respect to time. Equation 4 is generally true. Now we assume that now $\alpha = \omega_a t$, $\psi = \psi_0\cos\omega_p t$, ψ_0 is small and that $\omega_p \gg \omega_a$, so that $\Delta\psi$ is also small. Then

$$\begin{aligned} |\mathbf{s}|(\omega_a\cos\omega_a t - \sin\omega_a t(\dot{\Delta\psi})\sin\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\omega_a t(1-\frac{1}{2}\psi^2)\\ |\mathbf{s}|(\omega_a\cos\omega_a t - \sin\omega_a t(\dot{\Delta\psi})\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\omega_a t(1-\frac{1}{2}\psi_0^2\cos^2\omega_p t)\\ |\mathbf{s}|(\omega_a\cos\omega_a t - \sin\omega_a t(\dot{\Delta\psi})\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\omega_a t(1-\frac{1}{2}(\frac{1}{2}\psi_0^2(1-\cos 2\omega_p t))\\ |\mathbf{s}|(\omega_a\cos\omega_a t - \sin\omega_a t(\dot{\Delta\psi})\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\alpha(1-\frac{1}{2}(\frac{1}{2}\psi_0^2(1-\cos 2\omega_p t))\end{aligned}$$

The precession plane oscillates about the radial ($\hat{\boldsymbol{\rho}}$) direction many times (ω_p/ω_a times) per period of rotation of polarization about $\hat{\mathbf{z}}$. We take the average of the rapidly changing contribution (namely zero) leaving

$$\begin{aligned} |\mathbf{s}|(\omega_a\cos\omega_a t) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\omega_a t(1-\frac{1}{2}(\frac{1}{2}\psi_0^2)\\ \rightarrow \omega_a &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|(1-\frac{1}{4}\psi_0^2)\end{aligned}$$

In order to get some sense of $\Delta\psi$ let's try integrating Equation 3. We assume that $\alpha = \omega_a t$ which will not be true in general, but in any event we measure an average ω_a.

$$\begin{aligned}
|\mathbf{s}|\frac{d}{dt}(\sin\alpha\cos(\Delta\psi)) &= -\frac{e}{mc}a_\mu B|\mathbf{s}|\cos\alpha\cos\psi \\
\sin\omega_a t\cos(\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\int\cos\omega_a t(1-\frac{1}{4}\psi_0^2(1-\cos 2\omega_p t))dt \\
\sin\omega_a t\cos(\Delta\psi) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\left[\frac{\sin\omega_a t}{\omega_a}(1-\frac{1}{4}\psi_0^2)+\frac{\psi_0^2}{4}\int(\cos\omega_a t\cos 2\omega_p t)dt\right] \\
\sin\omega_a t(1-\frac{1}{2}(\Delta\psi)^2) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\left[\frac{\sin\omega_a t}{\omega_a}(1-\frac{1}{4}\psi_0^2)+\frac{\psi_0^2}{4}\int(\cos\omega_a t\cos 2\omega_p t)dt\right] \\
1-\frac{1}{2}(\Delta\psi)^2 &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\left[\frac{1}{\omega_a}(1-\frac{1}{4}\psi_0^2)+\frac{\psi_0^2}{4\sin\omega_a t}\int(\cos\omega_a t\cos 2\omega_p t)dt\right] \\
\omega_a(1-\frac{1}{2}(\Delta\psi)^2) &\sim -\frac{e}{mc}a_\mu B|\mathbf{s}|\left[\frac{1}{\omega_a}(1-\frac{1}{4}\psi_0^2)+\frac{\psi_0^2}{4\sin\omega_a t}\int(\cos\omega_a t\cos 2\omega_p t)dt\right]
\end{aligned} \tag{5}$$

For long times, and as long as $\omega_a \neq 2\omega_p$, the integral (second term in brackets on the right) sums to zero and then $\Delta\psi$ is zero. Near resonance $\omega_a \sim 2\omega_p$ the approximation breaks down. Presumably the plane of polarization is flipped about the $\hat{\boldsymbol{\rho}}$ direction.

Efield

$$\begin{aligned}
\frac{d}{dt}(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}) &= -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[a_\mu\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(a_\mu-\frac{m^2c^2}{p^2}\right)\beta\mathbf{E}\right] \\
&\sim -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[a_\mu\hat{\boldsymbol{\beta}}\times\mathbf{B}+\left(a_\mu-\frac{m^2c^2}{p_m^2}(1-2\frac{\Delta p}{p_m})\right)\beta\mathbf{E}\right] \\
&\sim -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[a_\mu\hat{\boldsymbol{\beta}}\times\mathbf{B}-2a_\mu\frac{\Delta p}{p_m}\beta\mathbf{E}\right]
\end{aligned}$$

Substitution of the same unperturbed solution yields the efield correction

$$\begin{aligned}
\Delta\omega_E &= -\frac{e}{mc}a_\mu(-2\frac{\Delta p}{p}\beta E_\rho) \\
\rightarrow \frac{\Delta\omega_E}{\omega_a}=C_e &= \frac{-2\langle\frac{\Delta p}{p}\beta E_\rho\rangle}{B}
\end{aligned}$$

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