Efield and Pitch Systematics

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CONTENTS

I. PRECESSION

Our unperturbed system is a muon precessing in a uniform magnetic field with zero electric field. The rate of change of the polarization with respect to the muon velocity, the thing that we measure, is described by Jackson 11.171

$$\frac{d}{dt}(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}) = -\frac{e}{mc}\mathbf{s}_\perp\cdot\left[a_\mu\hat{\boldsymbol{\beta}}\times\mathbf{B} + \left(a_\mu - \frac{m^2c^2}{p^2}\right)\beta\mathbf{E}\right] \tag{1}$$

We use a cylindrical coordinate system where $\mathbf{B}=B\hat{\mathbf{z}}$, $\hat{\boldsymbol{\rho}}$ is the radial direction and $\hat{\boldsymbol{\phi}}$ is the azimuthal direction. Our trial solution for $\mathbf{s}(t)$ in the ρ,ϕ plane is $\mathbf{s}(t) = |\mathbf{s}|(\cos\omega_a t\hat{\boldsymbol{\rho}} - \sin\omega_a t\hat{\boldsymbol{\phi}})$ where in the unperturbed system $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\phi}}$. Then

$$\mathbf{s}_\perp = \mathbf{s} - (\hat{\boldsymbol{\beta}}\cdot\mathbf{s})\cdot\hat{\boldsymbol{\beta}} = |\mathbf{s}|\cos\omega_a t\hat{\boldsymbol{\rho}}$$

and

$$\hat{\boldsymbol{\beta}}\cdot\mathbf{s} = -|\mathbf{s}|\sin\omega_a t$$

Substitution into 1 gives us

$$\omega_a = \frac{e}{mc}Ba_\mu \tag{2}$$

It should be clear from Equation 1 that in general, only the radial component of $\mathbf{s}_\perp$ will contribute, since the magnetic field is in the $\hat{\mathbf{z}}$ direction and the velocity always in the $\phi - z$ plane so that the cross product is always exclusively in the radial direction. With that in mind we will take

$$\mathbf{s}_\perp = |\mathbf{s}|\cos\omega_a t\hat{\boldsymbol{\rho}}$$

as the form for $\mathbf{s}_\perp$ as long as the velocity is in the $\phi - z$ plane. That is a convenient starting point as it allows integration of 1. For example, when $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\phi}}$ and $\mathbf{E}=0$, we can integrate 1

$$\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\beta}} = \int -\frac{e}{mc}a_\mu\hat{\mathbf{s}}_\perp\cdot(\hat{\boldsymbol{\beta}}\times\mathbf{B})dt = -\frac{e}{mc}a_\mu B\frac{\sin\omega_m t}{\omega_m} = -\omega_a\frac{\sin\omega_m t}{\omega_m}$$

Recall

$$\hat{\mathbf{s}}_\perp = \hat{\mathbf{s}} - (\hat{\boldsymbol{\beta}}\cdot\hat{\mathbf{s}})\hat{\boldsymbol{\beta}} = \cos\omega_a t\hat{\boldsymbol{\rho}}$$
$$\hat{\mathbf{s}}_\parallel = (\hat{\boldsymbol{\beta}}\cdot\hat{\mathbf{s}})\hat{\boldsymbol{\beta}} = -\frac{\omega_a}{\omega_m}\sin\omega_m t\hat{\boldsymbol{\beta}}$$

Since $\hat{\mathbf{s}} = \hat{\mathbf{s}}_\perp + \hat{\mathbf{s}}_\parallel$, and $|\hat{\mathbf{s}}| = 1$, it follows that $\hat{\mathbf{s}}_\perp^2 + \hat{\mathbf{s}}_\parallel^2 = \cos^2\omega_m t + \left(\frac{\omega_a}{\omega_m}\right)^2 \sin\omega_m^2 t = 1 \rightarrow \omega_m = \omega_a$.

II. PITCH

A. Time independent pitch

$$\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\beta}} = \int -\frac{e}{mc}a_\mu B\cos\omega_m t\cos\psi dt \tag{3}$$

If ψ is independent of time then

$$\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\beta}} = -\omega_a\frac{\sin\omega_m t}{\omega_m}\cos\psi$$

and evidently has sinusoidal time dependence with frequency ω_m.

$$-\sin\omega_m t = -\omega_a\frac{\sin\omega_m t}{\omega_m}\cos\psi dt$$
$$\rightarrow \omega_m = \omega_a\cos\psi$$

where ω_a is defined in Equation 2 and ω_m is what we measure.

We note that the component of the polarization vector in the $\rho-\phi$ plane rotates about the z-axis. The component perpendicular to that plane is fixed. And if there happens to be a component of velocity that is parallel to the magnetic field, the polarization vector will continue rotating about the z-axis, confined to the $\rho-\phi$ plane.

B. Pitching motion

Now suppose the muon is executing vertical betatron oscillations in the $\phi-z$ plane. Then

$$\hat{\boldsymbol{\beta}} = \cos\psi(t)\hat{\boldsymbol{\phi}} + \sin\psi(t)\hat{\mathbf{z}} \tag{4}$$

where $\psi(t)$ is the angle of the trajectory with respect to the $\rho-\phi$ plane. (See Section VI for more discussion of Equation 4.) If we imagine for a moment that there is no magnetic field, the muon will execute betatron oscillations in the focusing field of the quadrupoles. At the magic momentum, absent a magnetic field, the orientation of the polarization of the magic momentum muon with respect to the velocity is frozen. That is because

$$\frac{d}{dt}(\hat{\boldsymbol{\beta}}\cdot\mathbf{s}) = 0.$$

Next restore the uniform and time independent magnetic field and assume free betatron oscillations so that $\psi(t) = \psi_0\cos\omega_p t$. Let's imagine that at time t, $\psi(t) = 0$ ($\hat{\boldsymbol{\beta}}\cdot\mathbf{B} = 0$), and the polarization $\mathbf{s}$ is in the $\rho-\phi$ plane. If the E-field dominates ($\omega_p \gg \omega_a$) then as the pitch angle increases, the polarization will evolve a z-component that follows the z-component of velocity. If the magnetic field dominates ($\omega_a \gg \omega_p$), the polarization will be fixed in the $\rho-\phi$ plane. More generally, in the regime where $\omega_p \gg \omega_a$, the effect of the magnetic field is that the z-component of the polarization lags (or leads) the z-component of the velocity. The plane of the polarization vector will oscillate about the $\hat{\boldsymbol{\rho}}$ axis. Returning to Equation 3

$$\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\beta}} = \omega_a\int\cos\omega_m t\cos\psi(t)dt \tag{5}$$

In our experiment the frequency of vertical oscillations is much greater than the precession frequency, and since we measure the average precession frequency, rather than any component with the time dependence of the vertical oscillations, we average over that shorter time scale. Then

$$\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\beta}} = -\omega_a \int \cos \omega_m t \langle \cos \psi(t) \rangle dt$$
$$-\sin \omega_m t = -\omega_a \frac{\sin \omega_m t}{\omega_m} \langle \cos \psi(t) \rangle dt$$

and where $|\psi| \ll 1$,

$$\sin \omega_m t = \omega_a \frac{\sin \omega_m t}{\omega_m} (1 - \frac{1}{2} \langle \psi^2 \rangle)$$
$$\rightarrow \omega_m = \omega_a (1 - \frac{1}{2} \langle \psi^2 \rangle) \tag{6}$$
$$\tag{7}$$

By averaging over the pitching time scale, we are ignoring the z-component of polarization that lags or leads the z-component of velocity. We can determine this component iteratively. For our first iteration we assume that $\mathbf{s}_\perp = |\mathbf{s}| \cos \omega_m t \hat{\boldsymbol{\rho}}$ as before, so that at $t = 0$ the polarization is initially in the $\hat{\boldsymbol{\rho}}$ direction. Substitute the explict form for $\psi(t)$ into Equation 5, expand about small $|\psi|$ and integrate

$$\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\beta}} = -\omega_a \int \cos \omega_m t (1 - \frac{1}{4} \psi_0^2 (1 + \cos 2\omega_p t)) dt$$
$$= -\omega_a \left[\frac{\sin \omega_m t}{\omega_m} (1 - \frac{1}{4} \psi_0^2) - \frac{1}{8} \psi_0^2 \left(\frac{\sin(\omega_m + 2\omega_p)t}{\omega_m + 2\omega_p} + \frac{\sin(\omega_m - 2\omega_p)t}{\omega_m - 2\omega_p} \right) \right] \tag{8}$$

We have used $\langle \psi^2 \rangle = \langle (\psi_0 \cos \omega_p t)^2 \rangle = \frac{1}{2} \psi_0^2$ so that $\omega_m = \omega_a (1 - \frac{1}{4} \psi_0^2)$ equivalent to Equation 6. Note that as anticipated, the second term in Equation 8 averages to zero on the pitching time scale.

The contribution of the pitching motion to the average precession frequency is

$$\frac{\Delta \omega_{pitch}}{\omega_a} = C_p = -\frac{1}{4} \psi_0^2$$

C. Measurement

The vertical betatron pitching motion, assuming linear fields and continuous quadrupoles, is characterized by

$$y = a \sqrt{\beta_z} \sin(\omega_p t)$$
$$\psi = \frac{a}{\sqrt{\beta_z}} \cos(\omega_p t) = \psi_0 \cos \omega_p t$$

Then

$$\psi_0^2 = 2 \langle y^2 \rangle / \beta_z^2 = 2 \frac{n}{R_0^2} \langle y^2 \rangle$$

and

$$\langle C_p \rangle = -\frac{n}{2} \frac{\langle y^2 \rangle}{R_0^2}$$

where $\beta_z = R_0 / n$

D. High frequency component

Equation 8 gives $\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\beta}} = \frac{1}{\beta}(s_z \beta_z + s_\phi \beta_\phi)$. We argue above that at the magic momentum, and when $\omega_p \gg \omega_a$, the component of the spin vector in the z-direction will simply follow the z-component of velocity. With that approximation, the polarization

$$\mathbf{s}_0 = |\mathbf{s}| [\cos \omega_a t \hat{\boldsymbol{\rho}} - \sin \omega_a t (\cos \psi \hat{\boldsymbol{\phi}} + \sin \psi \hat{\mathbf{z}})] \tag{9}$$

That is, the polarization vector is rotating about a comoving axis ($\mathbf{z}'$), that is parallel to $\hat{\boldsymbol{\beta}} \times \hat{\boldsymbol{\rho}}$. Then we can write

$$\hat{\beta}_z s_z = \beta_z s_{z0} + \Delta[\beta_z s_z] \tag{10}$$

where

$$\hat{\beta}_z s_{z0} = \frac{\beta_z}{\beta} s_{z0} = (\psi_0 \cos\omega_p t)[(\sin\omega_a t)\sin(\psi_0 \cos\omega_p t)] \tag{11}$$

using $\beta_z/\beta = \psi_0 \cos\omega_p t$, and $\Delta[\beta_z s_z]$ is the contribution from that part of s_z that does not follow β_z, namely the high frequency part of Equation 8.

$$\Delta[\beta_z s_z] = -\frac{1}{4}\psi_0^2 \frac{\omega_a}{2}\left(\frac{\sin(\omega_m + 2\omega_p)t}{\omega_m + 2\omega_p} + \frac{\sin(\omega_m - 2\omega_p)t}{\omega_m - 2\omega_p}\right) \tag{12}$$

Note that Equation 10 is not an exact solution since in order to calculate it we assumed that $\mathbf{s}_\perp = |\mathbf{s}| \cos\omega_m t \hat{\boldsymbol{\rho}}$. The exact solution will satisfy $(\mathbf{s}_\parallel)^2 + (\mathbf{s}_\perp)^2 = |\mathbf{s}| = 1$. We can use this normalization relationship to solve for the next order correction to $\mathbf{s}_\perp$, that is

$$\mathbf{s}'_\perp = \left(1 - (\hat{\boldsymbol{\beta}} \cdot \hat{\mathbf{s}})^2\right)^{1/2} \tag{13}$$

and then substitute $\mathbf{s'}_\perp$ back into Equation 5 and integrate again. We can of course compute the exact time evolution of the polarization by integrating the Thomas-BMT equation. The exact solution is compared to Equations 11 and 12 in Figure 1. At $t = 0$ in the simulation, the magic momentum muon is polarized in the $\hat{\boldsymbol{\rho}}$ direction. The intial velocity is $\beta = (p_m/mc)(\cos\psi\hat{\boldsymbol{\phi}} + \sin\psi\hat{\mathbf{z}})$ where $\psi = 2$ mrad. The initial coordinates are on the magic orbit. The muon is tracked through the fields of the ring. The polarization is computed by integration of the Thomas-BMT equation.

E. Next order

We substitute $\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\beta}}$ from 8 into 13 to determine the first order change in magnitude of $\mathbf{s}_\perp$.

$$s'_\perp \sim |\mathbf{s}| \left[\cos\omega_m t - \frac{1}{8}\psi_0^2 \sin\omega_m t\,(\text{stuff}) + \frac{1}{2}\left(\frac{\psi_o^2\omega_a}{8}\right)^2 (\text{stuff})^2\right]$$

Then we can use that in 1 to compute the next higher order contribution in ψ_0^2

III. EFIELD

The contribution of the electric field to the precession frequency is more straightforward. Expand about the magic momentum in Equation 1

$$\begin{aligned}
\frac{d}{dt}(\hat{\boldsymbol{\beta}} \cdot \mathbf{s}) &= -\frac{e}{mc}\mathbf{s}_\perp \cdot \left[a_\mu \hat{\boldsymbol{\beta}} \times \mathbf{B} + \left(a_\mu - \frac{m^2c^2}{p^2}\right)\beta\mathbf{E}\right] \\
&\sim -\frac{e}{mc}\mathbf{s}_\perp \cdot \left[a_\mu \hat{\boldsymbol{\beta}} \times \mathbf{B} + \left(a_\mu - \frac{m^2c^2}{p_m^2}(1 - 2\frac{\Delta p}{p_m})\right)\beta\mathbf{E}\right] \\
&\sim -\frac{e}{mc}\mathbf{s}_\perp \cdot \left[a_\mu \hat{\boldsymbol{\beta}} \times \mathbf{B} - 2a_\mu \frac{\Delta p}{p_m}\beta\mathbf{E}\right]
\end{aligned}$$

With the usual substitution of $\mathbf{s}_\perp = |\mathbf{s}| \cos\omega_m t \hat{\boldsymbol{\rho}}$ we have

$$\Delta\omega_E = -\frac{e}{mc}a_\mu(-2\frac{\Delta p}{p}\beta E_\rho)$$

$$\rightarrow \frac{\Delta\omega_E}{\omega_a} = C_e = \frac{-2\langle\frac{\Delta p}{p}\beta E_\rho\rangle}{B}$$

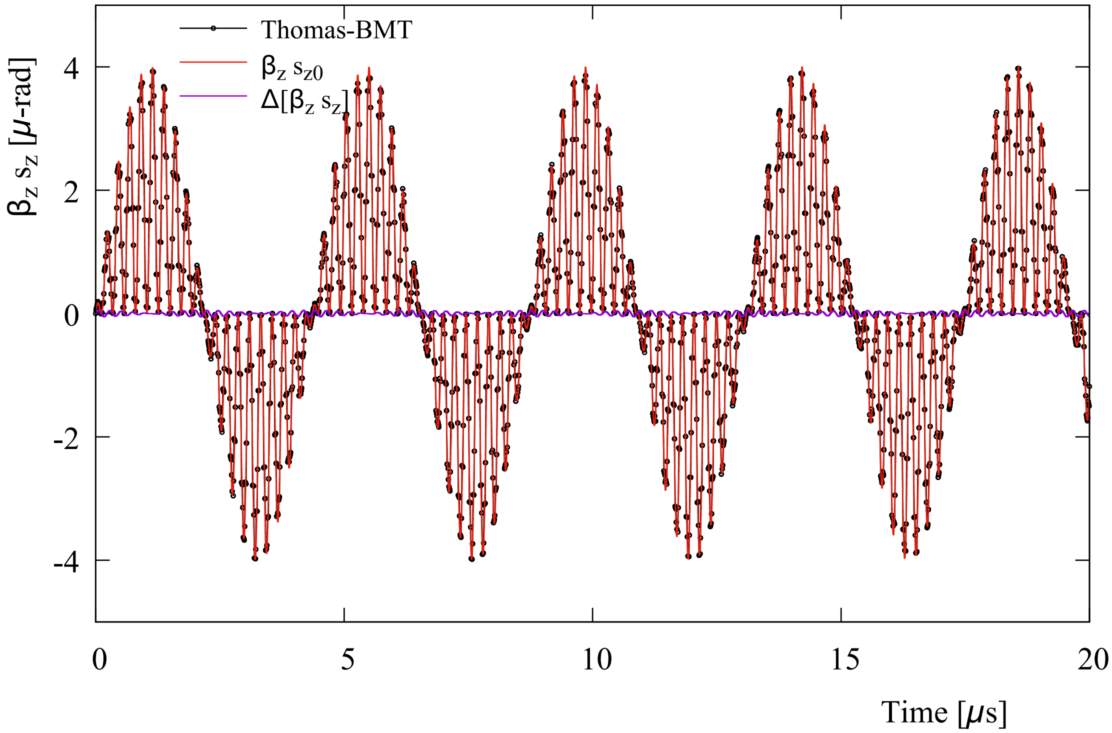

FIG. 1: The projection of spin on the z-component of velocity. The black curve with points is from iteration of the Thomas-BMT equation. The red line ('$\beta_z s_{z0}$'), Equation 11, is the projection of polarization onto the z-axis with the approximation that the polarization is given by Equation 9. The purple line is the correction term ($\Delta[\beta_z s_z]$), Equation 12. We see that our approximation is in very good agreement with the exact result. Note also that the contribution of the z-component to the $\hat{\boldsymbol{\beta}} \cdot \mathbf{s}$ is at the level of a few parts per millon and that the correction $\Delta[\beta_z s_z]$ two orders of magnitude smaller.

We can write $\frac{Deltap}{p} = x_e \eta$, where x_e is the momentum dependent displacement of the closed orbit from the magic radius. The radial electric field is given by

$$E_\rho = n \left(\frac{\beta B}{R_0} \right) x_e$$

so that

$$\langle C_e \rangle = -2\beta^2 n(1-n) \frac{\langle x_e^2 \rangle}{R_0^2}$$

where we have used $\eta = R_0/(1-n)$.

IV. EXTRACTING PITCH AND EFIELDS CORRECTIONS FROM THE DIFFERENCE EQUATION

Consider the familiar expression for the difference of the precession frequency and the cyclotron frequency.

$$\boldsymbol{\omega}_{diff} = -\frac{e}{m} \left[a_\mu \mathbf{B} - a_\mu \left(\frac{\gamma}{\gamma+1} \right) (\boldsymbol{\beta} \cdot \mathbf{B}) \boldsymbol{\beta} - \left(a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\boldsymbol{\beta} \times \mathbf{E}}{c} \right] \tag{14}$$

In our experiment we measure the projection of the polarization onto the velocity. We measure that part of $\boldsymbol{\omega}_{diff}$, ($\omega_a$) that is perpendicular to the velocity. We are quite insensitive to rotations about the direction of motion. We measure $|\boldsymbol{\omega}_\perp| = |\hat{\boldsymbol{\beta}} \times \boldsymbol{\omega}_{diff}|$.

Then in the absence of an electric field,

$$\omega_a = |\boldsymbol{\omega}_\perp| = -\frac{e}{m} a_\mu |\hat{\boldsymbol{\beta}} \times \left[\mathbf{B} - \left(\frac{\gamma}{\gamma+1} \right) (\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} \right] |$$
$$= -\frac{e}{m} a_\mu |\hat{\boldsymbol{\beta}} \times \mathbf{B}|$$

Note that the second term on the right of Equation 14 is parallel to the velocity and therefore does not contribute to ω_a.

Then it follows that

$$\phi_a(T) = -\frac{e}{m} a_\mu \int^T |(\hat{\boldsymbol{\beta}} \times \mathbf{B})| dt$$

where $\omega_a = \frac{d\phi_a}{dt}$.

V. ELECTRIC FIELD CORRECTION FROM THE DIFFERENCE EQUATION

Consider next the contribution from the electric field to $\boldsymbol{\omega}_{perp}$. We see that the electric field contribution is exclusively perpendicular to the velocity. Therefore

$$\Delta\omega_a = \Delta\boldsymbol{\omega}_\perp = -\left(a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\boldsymbol{\beta} \times \mathbf{E}}{c}$$

Then

$$\Delta\phi_a(\text{Efield}) = -\frac{e}{m} \left(a_\mu - \frac{1}{\gamma^2-1} \right) \int^T \frac{\boldsymbol{\beta}\times\mathbf{E}}{c} dt$$

where $\Delta\omega_a = \frac{d\phi_a(Efield)}{dt}$.

We can write equivalently

$$\Delta\phi_a(\text{Efield}) = -\frac{e}{m} \left(a_\mu - \frac{1}{(E/m)^2-1} \right) \int^T \frac{\boldsymbol{\beta}\times\mathbf{E}}{c} dt$$
$$= -\frac{e}{m} \left(a_\mu - \frac{m^2c^2}{p^2} \right) \int^T \frac{\boldsymbol{\beta}\times\mathbf{E}}{c} dt$$
$$= -\frac{e}{m} \left(a_\mu - \frac{m^2c^2}{p_m^2}(1 - 2\frac{\Delta p}{p_m}) \right) \int^T \frac{\boldsymbol{\beta}\times\mathbf{E}}{c} dt$$
$$= -2\frac{e}{m} a_\mu \frac{\Delta p}{p_m} \int^T \frac{\boldsymbol{\beta}\times\mathbf{E}}{c} dt$$

where at the magic momentum p_m, $a_\mu = \frac{m^2c^2}{p_m}^2$ and $\Delta p = p - p_m$.

VI. PITCHING VELOCITY

The muons oscillate vertically in the electric field of the quadrupoles. In the continuous quad approximation the time dependence of the pitch angle is

$$\psi(t) = \psi_0 \cos(\omega_p t)$$

where ω_p is the vertical betatron frequency. The momentum p_ϕ in the azimuthal direction is constant, that is, independent of the pitch angle ψ. The force on the muons due to the electric field of the quadrupoles is in the vertical direction. The pitch angle is related to momenta according to $p_z/p_\phi = \tan\psi$. Since the muon gains and loses energy to the electric field of the quadrupole, the magnitude of the energy and velocity $|\hat{\boldsymbol{\beta}}|$ will, like the z-component of the momentum, be time dependent. We can write the azimuthal and vertical components of the velocity as

$$\beta_\phi = \frac{p_\phi}{E}$$
$$\beta_z = \frac{p_z}{E}$$

Then

$$\begin{aligned}\hat{\boldsymbol{\beta}}_\phi &= \frac{\beta_\phi}{[\beta_\phi^2+\beta_z^2]^{\frac{1}{2}}} \\ &= \frac{p_\phi/E}{[p_\phi^2/E^2+p_z^2/E^2]^{\frac{1}{2}}} \\ &= \frac{p_\phi}{p_\phi[1+p_z^2/p_\phi^2]^{\frac{1}{2}}} \\ &\sim (1-\frac{1}{2}\frac{p_z^2}{p_\phi^2}) = 1-\frac{1}{2}\tan^2\psi \end{aligned} \tag{15}$$

Also

$$\begin{aligned}\hat{\boldsymbol{\beta}}_z &= \frac{\beta_z}{[\beta_\phi^2+\beta_z^2]^{\frac{1}{2}}} \\ &= \frac{p_z/E}{[p_\phi^2/E^2+p_z^2/E^2]^{\frac{1}{2}}} \\ &= \frac{p_z}{p_\phi[1+p_z^2/p_\phi^2]^{\frac{1}{2}}} \\ &\sim \frac{p_z}{p_\phi}(1-\frac{1}{2}\frac{p_z^2}{p_\phi^2}) = \tan\psi(1-\frac{1}{2}\tan^2\psi) \end{aligned} \tag{16}$$

We might use this more precise form for $\hat{\boldsymbol{\beta}}$ in Equation 3 (rather than what is given in Equation 4). However, to order ψ^2 the result is the same. That is

$$\hat{\boldsymbol{\beta}}_\phi = 1-\frac{1}{2}\tan^2\psi \sim 1-\frac{1}{2}\psi^2 \sim \cos\psi$$

and

$$\hat{\boldsymbol{\beta}}_z = \tan\psi(1-\frac{1}{2}\tan^2\psi) \sim \sin\psi$$

Recall that $\hat{\boldsymbol{\beta}}$ appears in the integral in the cross product with $\mathbf{B}$ and $\mathbf{B} = B\hat{\mathbf{z}}$.

$$\begin{aligned}\hat{\boldsymbol{\beta}}\times\mathbf{B} &= \hat{\boldsymbol{\beta}}_\phi B \\ &\sim B(1-\frac{1}{2}\tan^2\psi) \sim B(1-\frac{1}{2}\psi^2)\end{aligned}$$

Since

$$\cos\psi \sim 1-\frac{1}{2}\psi^2,$$

to order ψ^2, $|\hat{\boldsymbol{\beta}}\times\mathbf{B}| \sim \cos\psi$, and consistent with the approximations used throughout. That is because $\cos\psi \sim 1-\frac{1}{2}\tan^2\psi$ for small ψ. The near equivalence is evident in a comparsion (see Fig 2) of $\boldsymbol{\beta}$ derived from integration of the equations of motion and the formula used above, namely

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= \cos(\psi(t))\hat{\boldsymbol{\phi}} + \sin(\psi(t))\hat{\mathbf{z}} \\ &= \cos(\psi_0\cos\omega_p t)\hat{\boldsymbol{\phi}} + \sin(\psi_0\cos(\omega_p t))\hat{\mathbf{z}}\end{aligned} \tag{17}$$

Note that if the focusing were magnetic rather than electric, the total momentum would be constant and independent of pitch in which case

$$\begin{aligned} p_\phi^2 + p_z^2 &= p^2 \\ p_\phi^2/p^2 + p_z^2/p^2 &= 1 \\ \rightarrow (\beta_\phi/\beta)^2 + (\beta_z/\beta)^2 &= 1\end{aligned}$$

But we see that for $\psi \ll 1$ the time dependence of $\hat{\boldsymbol{\beta}}$ is the same for electric and magnetic focusing. Note also that the second order dependence of azimuthal momentum on pitch angle is typically ignored in the first order solution of the equations of motion for the weak focusing storage ring.

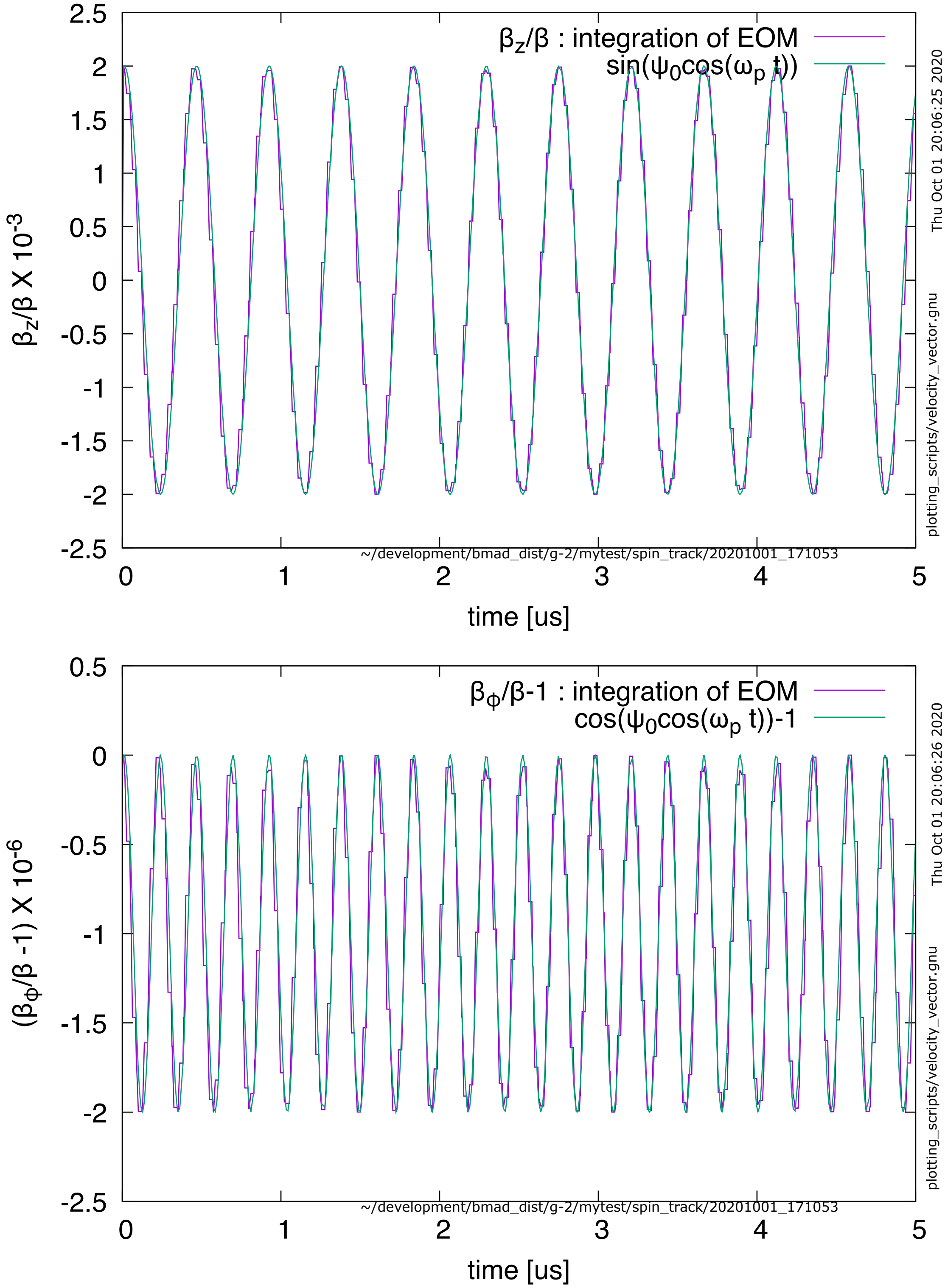

FIG. 2: β_z (top) and β_ϕ (bottom) are determined by integration of the equations of motion throught a model of the g-2 ring with discrete quadrupoles and equivalent field index $n = 0.1058$. The initial coordinates of the trajectory are on the magic orbit ($\rho = 7.112$ m, $z = 0$ and $\psi = 0.002$). Equation 17 is in good agreement with the 'exact' result.