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# Quantum Mechanical Evaluation of Spin Dynamics in the g-2 Storage Ring

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## Abstract

The evolution of the muon spin in the g-2 storage ring is characterized by the Thomas-BMT equation. The equation gives the time dependence of the orientation of the muon spin on the magnetic and electric fields and the muon velocity. That interaction can alternatively be described by an effective Hamiltonian, and the dynamics by Schrodinger's equation. In the following, an effective Hamiltonian is derived for case of vertical oscillations driven by electrostatic focusing (pitching motion), longitudinal magnetic fields, and radial magnetic fields. The formalism yields exact solutions for the muon precession frequency in the rotating frame in the case of a fixed pitch angle (zero electric field), and uniform radial and longitudinal fields. The effect of vertical oscillations including electrostatic focusing is determined perturbatively, as is the effect of non uniform longitudinal fields.

**CONTENTS**

## I. INTRODUCTION

In the g-2 experiment we measure the time dependence of the projection of the muon spin along its direction of motion, $\hat{\boldsymbol{\beta}}\cdot\mathbf{s}$. The muon velocity ($\boldsymbol{\beta}$) is determined by the Lorentz force law, and the evolution of the spin ($\mathbf{s}$) by the Thomas-BMT equation. In the Thomas-BMT equation the interaction of the magnetic moment of the muon with the electric and magnetic fields in the muon storage ring, appears as a torque $\boldsymbol{\mu}_{eff}\times\mathbf{B}_{eff}$, where $\boldsymbol{\mu}_{eff}$ is proportional to the magnetic moment and $\mathbf{B}_{eff}$ depends on the fields and the velocity. Alternatively, and equivalently, the interaction can be described by a hamiltonian $\mathcal{H}=-\boldsymbol{\mu}_{eff}\cdot\mathbf{B}_{eff}$. Then the Schrodinger equation determines the evolution of the spin. The solution to the Schrodinger equation with Hamiltonian $\mathcal{H}$ is thus equivalent to the solution of the Thomas equation. The advantage of the Hamiltonian formulation is that a) in the case of uniform electric and magnetic fields, exact solutions exist and share a common structure, and b) in the event of time dependent fields for which there is no exact solution, there is a well established procedure for finding approximate solutions, using time dependent perturbation theory.

In section II we review the quantum mechanics of a muon at rest in a time dependent magnetic field. Section III describes the transformation to a rotating frame. The Hamiltonian corresponding to the dynamics of the Thomas-BMT equation is derived in V. Section VI is devoted to analysis of the effect of vertical motion on the precession frequency, beginning with the exact solution for fixed pitch (fixed vertical angle) and proceeding to a perturbative evaluation for the muon oscillating in a vertically focusing electrostatic field with arbitrary frequency. The effect of a longitudinal magnetic field on the evolution of the spin is explored in Section VIII, with an exact solution in the case of a uniform field and perturbatively in the case that the longitudinal field depends on azimuth. The solution for a uniform radial field is given in Section VII, and for an off momentum muon on its closed orbit (electric field correction) in Section IX. And just for fun we consider the evolution of the spin of a muon experiencing electrostatic vertical focusing in the absence of a magnetic field and explore its momentum dependence in Section X. Section XI summarizes effects due to pitch, radial and longitudinal fields, and electric field.

## II. MUON IN A TIME VARYING MAGNETIC FIELD

We will begin by reviewing the dynamics of a muon at rest in a time varying field, and then adapt that well known formalism to a muon in motion in a fixed field[1][p. 330] (*see Griffiths Quantum Mechanics 1st edition, problem 9.7 on page 305 and section 10.1.3 page 330).* The time dependence of the spin of a muon at rest in a magnetic field is given by

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\mu} \times \mathbf{B}$$

where $\boldsymbol{\mu} = \frac{ge}{2mc}\mathbf{s}$. The system can equivalently be described in terms of an interaction energy $\mathcal{H} = -\boldsymbol{\mu} \cdot \mathbf{B}$. The Hamiltonian is

$$\mathcal{H} = -\boldsymbol{\mu} \cdot \mathbf{B} = -\frac{ge}{2mc}\frac{\hbar}{2}\boldsymbol{\sigma} \cdot \mathbf{B} = -\frac{ge}{2mc}\frac{\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Suppose that the z-component of the magnetic field is fixed and that the magnetic field transverse to the z-axis rotates about that axis with frequency $\omega$ (see Figure 1), so that

$$\mathbf{B} = B_z\hat{\mathbf{z}} + B_{xy}(\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}})$$

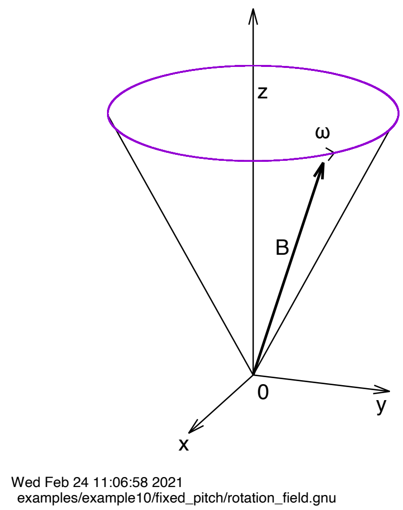


FIG. 1: The z-component of the magnetic field is fixed. The transverse component of the field rotates about the z-axis with frequency $\omega$

Then

$$\mathcal{H} = -\frac{ge}{2mc}\frac{\hbar}{2}\begin{pmatrix} B_z & B_{xy}e^{i\omega t} \\ B_{xy}e^{-i\omega t} & -B_z \end{pmatrix} \equiv -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & -\omega_0 \end{pmatrix} \tag{1}$$

where $\omega_0 = \frac{ge}{2mc}B_z$ and $\Omega = \frac{ge}{2mc}B_{xy}$. The time evolution is governed by Schrodinger's equation.

$$i\hbar\frac{\partial}{\partial t}\chi = \mathcal{H}\chi$$

For a spinor $\chi(t) = (a(t), b(t))$, we can write:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = \frac{i}{2}\begin{pmatrix} a(t)\omega_0 + b(t)\Omega e^{i\omega t} \\ a(t)\Omega e^{-i\omega t} - b(t)\omega_0 \end{pmatrix}. \tag{3}$$

## III. ROTATING FRAME

It is convenient to transform to a 'rotating frame', in which the effective Hamiltonian is independent of time. Then the Schrodinger equation (2) can be solved practically by inspection (*See Griffiths Quantum Mechanics, 1st edition, problem 9.7, p. 305*). Define

$$R = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

Then

$$R\mathcal{H}R^{-1} = R\begin{pmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & -\omega_0 \end{pmatrix} R^{-1} = \begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix}$$

and the Schrodinger equation 2 can be written as

$$\begin{aligned}
i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} &= -R^{-1}\frac{1}{2}\begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix} R \begin{pmatrix} a \\ b \end{pmatrix} \\
\rightarrow iR\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix} R \begin{pmatrix} a \\ b \end{pmatrix} \\
i\frac{d}{dt}\left[R\begin{pmatrix} a \\ b \end{pmatrix}\right] - i\frac{dR}{dt}\begin{pmatrix} a \\ b \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix} R \begin{pmatrix} a \\ b \end{pmatrix} \\
i\frac{d}{dt}\left[R\begin{pmatrix} a \\ b \end{pmatrix}\right] - \frac{\omega}{2}\sigma_z R\begin{pmatrix} a \\ b \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} \omega_0 & \Omega \\ \Omega & -\omega_0 \end{pmatrix} R \begin{pmatrix} a \\ b \end{pmatrix} \\
i\hbar\frac{d}{dt}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} &= \mathcal{H}'\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & \Omega \\ \Omega & -\omega_0 + \omega \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}
\end{aligned}$$

where

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = R\begin{pmatrix} a \\ b \end{pmatrix}$$

In the rotating frame the Hamiltonian is independent of time and the characteristic frequencies are simply the eigenvalues.

$$\omega'^{\pm} = \pm\sqrt{(\omega_0 - \omega)^2 + \Omega^2}$$

We next find the eigenvectors, take appropriate linear combinations and transform from the rotating frame back to the fixed frame. See Appendix XII B for details of the linear algebra.

(Alternatively see Appendix XII A for direct solution of the coupled differential equations 3)

## IV. OBSERVABLES AND POLARIZATION

By whichever method we choose, we find the solution to 2

$$\psi(t) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \left(\cos(\omega' t/2) - \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) e^{i\frac{\omega}{2}t} & -\frac{i\Omega}{\omega'}\sin(\omega' t/2)e^{i\frac{\omega}{2}t} \\ -\frac{i\Omega}{\omega'}\sin(\omega' t/2)e^{-i\frac{\omega}{2}t} & \left(\cos(\omega' t/2) + \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) e^{-i\frac{\omega}{2}t} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{4}$$

where

$$\omega' = [(\omega_0 - \omega)^2 + \Omega^2]^{1/2} \tag{5}$$

and $a_0$ and $b_0$ define the initial spin state. The polarization (the observable),

$$\mathbf{s} = \frac{1}{2}\langle \psi \mid \boldsymbol{\sigma} \mid \psi \rangle = \frac{1}{2}\left((a^*b + b^*a)\hat{\mathbf{x}} + -i(a^*b - b^*a)\hat{\mathbf{y}} + (|a(t)|^2 - |b(t)|^2)\hat{\mathbf{z}}\right) \tag{6}$$

Since

$$\begin{aligned} a^* &= \left[\left(\cos(\omega' t/2) + \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) a_0^* + \frac{i\Omega}{\omega'}\sin(\omega' t/2)b_0^*\right] e^{-i\frac{\omega t}{2}} \\ b &= \left[-\frac{i\Omega}{\omega'}\sin(\omega' t/2)a_0 + \left(\cos(\omega' t/2) + \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) b_0\right] e^{-i\frac{\omega t}{2}} \end{aligned}$$

we can write

$$\begin{aligned} a^*b = \Bigg( & a_0^*b_0\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega-\omega_0)^2 a_0^*b_0 - \Omega^2 b_0^*a_0 - (\omega-\omega_0)\Omega(|a_0|^2 - |b_0|^2)]\sin^2(\omega' t/2) \\ & + \frac{i}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)\left(a_0^*((\omega-\omega_0)b_0 - \Omega a_0) - b_0((\omega_0-\omega)a_0^* - \Omega b_0^*)\right)\Bigg] e^{-i\omega t} \end{aligned}$$

If the intial state is $a_0 = b_0 = \frac{1}{\sqrt{2}}$ (an eigenket of $\sigma_x$), the intial state polarization is $s_x = 1$. Then

$$\begin{aligned} a^*b &= \frac{1}{2}\Bigg(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega-\omega_0)^2 - \Omega^2]\sin^2(\omega' t/2) \\ &\quad + \frac{i}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)\left(((\omega-\omega_0) - \Omega) - ((\omega_0-\omega) - \Omega)\right)\Bigg) e^{-i\omega t} \\ &= \frac{1}{2}\left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega-\omega_0)^2 - \Omega^2]\sin^2(\omega' t/2) + \frac{2i}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)(\omega-\omega_0)\right) e^{-i\omega t} \end{aligned}$$

The polarization

$$\langle s_x \rangle = \left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega-\omega_0)^2 - \Omega^2]\sin^2(\omega' t/2)\right)\cos(\omega t) + \left(\frac{2}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)(\omega-\omega_0)\right)\sin(\omega$$

$$\langle s_y \rangle = -\left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega-\omega_0)^2 - \Omega^2]\sin^2(\omega' t/2)\right)\sin(\omega t) + \left(\frac{2}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)(\omega-\omega_0)\right)\cos$$

$$\langle s_z \rangle = \frac{1}{2}(|a|^2 - |b|^2) = \frac{1}{2}\left[\cos^2(\omega' t/2) + \frac{(\omega-\omega_0+\Omega)^2}{\omega'^2}(\sin^2\omega' t/2) - \left(\cos^2(\omega' t/2) + \frac{(\omega-\omega_0-\Omega)^2}{\omega'^2}\sin^2(\omega' t\right.\right.$$

$$= \frac{(\omega-\omega_0)\Omega}{\omega'^2}\sin^2(\omega' t/2)$$

## V. EFFECTIVE HAMILTONIAN FOR THOMAS EQUATION

Of course we are not interested in the precession of muon at rest in a time varying magnetic field, but rather a muon in motion in a fixed field, as described by the Thomas-BMT[2][p. 559] equation.

$$\frac{d\mathbf{s}}{dt} = \frac{e}{mc}\mathbf{s} \times \left[(a_\mu + \frac{1}{\gamma})\mathbf{B} - a_\mu \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E}\right] \tag{8}$$

or equivalently

$$\mathcal{H} = -\tilde{\boldsymbol{\mu}}\cdot\mathcal{B}$$

where $\tilde{\boldsymbol{\mu}} = \frac{e}{mc}\mathbf{s}$ and $\mathcal{B}$ is an effective magnetic field,

$$\mathcal{B} = \left[(a_\mu + \frac{1}{\gamma})\mathbf{B} - a_\mu \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E}\right]$$

## VI. HAMILTONIAN FOR A MUON WITH A VERTICAL COMPONENT OF VELOCITY

### A. Fixed pitch angle (spiral)

Consider a muon in a uniform magnetic field with a component of velocity parallel to the field. We work in a fixed cartesian coordinate system with $\mathbf{B} = B_z\hat{\mathbf{z}}$. (Our coordinate system is shown in Figure 2.) According to the Lorentz force law[2][p. 559]

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m}[\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})]$$

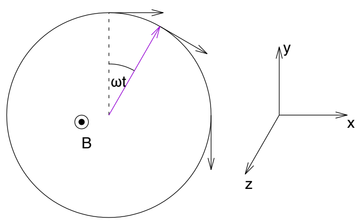


FIG. 2: The positive muon is injected at the top of the ring and rotates in the clockwise direction. The magnetic field is out of the page in the positive z-direction.

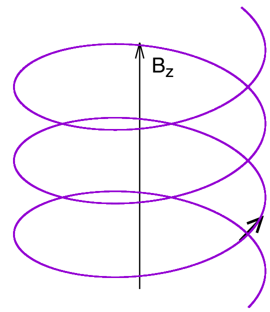


FIG. 3: Muon trajectory in a uniform magnetic field.

the particle trajectory is a clockwise spiral (see Figure 3), with velocity

$$\boldsymbol{\beta} = \beta_{xy}(\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}}) + \beta_z\hat{\mathbf{z}}.$$

(For the time being we assume that $\mathbf{E} = 0$.)

Then the effective magnetic field is

$$\begin{aligned}\mathcal{B} &= \left[(a_\mu + \frac{1}{\gamma})B_z\hat{\mathbf{z}} - a_\mu\frac{\gamma}{\gamma+1}(\beta_z B_z)\left(\beta_z\hat{\mathbf{z}} + \beta_{xy}(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\right)\right] \\ &= B_z\left[\left((a_\mu + \frac{1}{\gamma}) - a_\mu\frac{\gamma}{\gamma+1}\beta_z^2\right)\hat{\mathbf{z}} - \left(a_\mu\frac{\gamma}{\gamma+1}\beta_z\beta_{xy}(\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}}\right)\right]\end{aligned}$$

Define

$$\omega_0 \equiv \frac{e}{mc}B_z\left[a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\beta_z^2\right] \tag{9}$$

$$\Omega \equiv -\frac{e}{mc}B_z\left[a_\mu\frac{\gamma}{\gamma+1}\beta_z\beta_{xy}\right] \tag{10}$$

$$\omega = \omega_c = \frac{eB_z}{mc\gamma} \tag{11}$$

and our Hamiltonian assumes the form of Equation 1,

$$\mathcal{H} = -\tilde{\boldsymbol{\mu}}\cdot\mathcal{B} = -\frac{e}{mc}\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\mathcal{B} = -\frac{\hbar}{2}\begin{pmatrix}\omega_0 & \Omega e^{i\omega t}\\ \Omega e^{-i\omega t} & -\omega_0\end{pmatrix}. \tag{12}$$

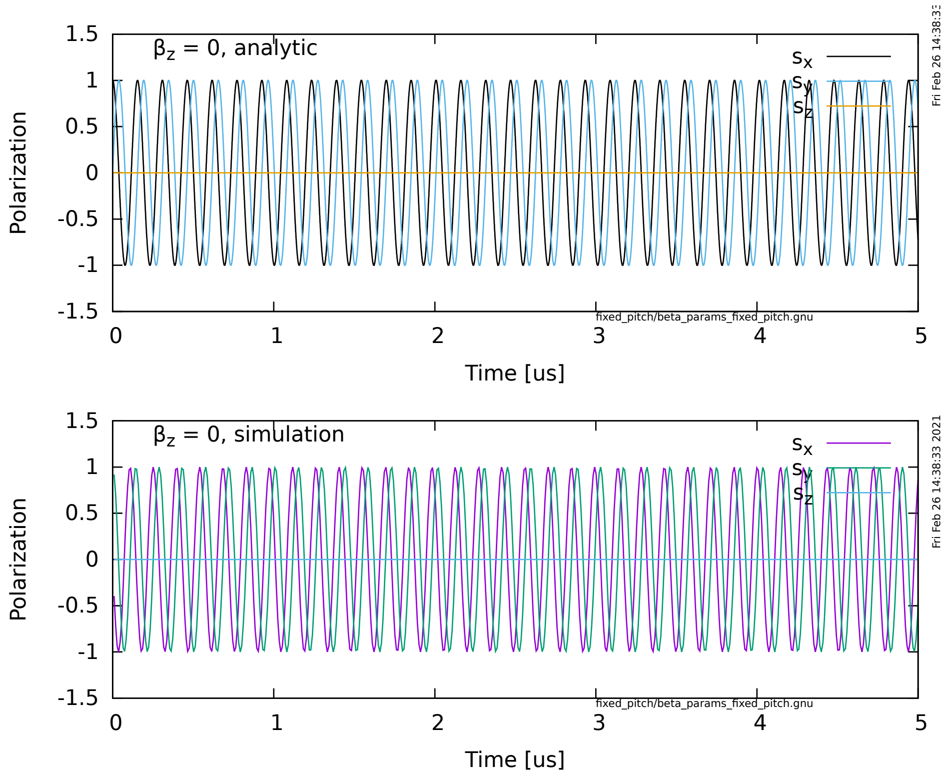


FIG. 4: Polarization of a muon following a circular trajectory, $\beta_z = 0$. The analytic solution (Equations 7) is on the top, and simulation (integration of the equations of motion and the BMT equation) at the bottom.

In the rotating frame

$$\mathcal{H}_{rot} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & \Omega \\ \Omega & -\omega_0 + \omega \end{pmatrix}$$

and the eigenvalue is the precession frequency in the rotating frame. Now we simply substitute 9 and 10 into Equations 7, to get the time dependence of the polarization.

The projection of the spin onto the direction of motion is given by

$$\begin{aligned} |\boldsymbol{\beta}|\hat{\boldsymbol{\beta}} \cdot s &= \left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega - \omega_0)^2 - \Omega^2]\sin^2(\omega' t/2)\right)\beta_{xy}(\cos^2(\omega t) + \sin^2(\omega t)) + \beta_z \frac{2(\omega - \omega_0)\Omega}{\omega'^2}\sin^2(\omega \\ &= \left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega - \omega_0)^2 - \Omega^2]\sin^2(\omega' t/2)\right)\beta_{xy} + \beta_z \frac{2(\omega - \omega_0)\Omega}{\omega'^2}\sin^2(\omega' t/2) \\ &= \frac{1}{2}\left(1 + \cos(\omega' t) - \frac{1}{\omega'^2}[(\omega - \omega_0)^2 - \Omega^2](1 - \cos\omega' t)\right)\beta_{xy} + \beta_z \frac{(\omega - \omega_0)\Omega}{\omega'^2}(1 - \cos(\omega' t)) \end{aligned}$$

The thing we measure is the time rate of change of the projection of the polarization on the

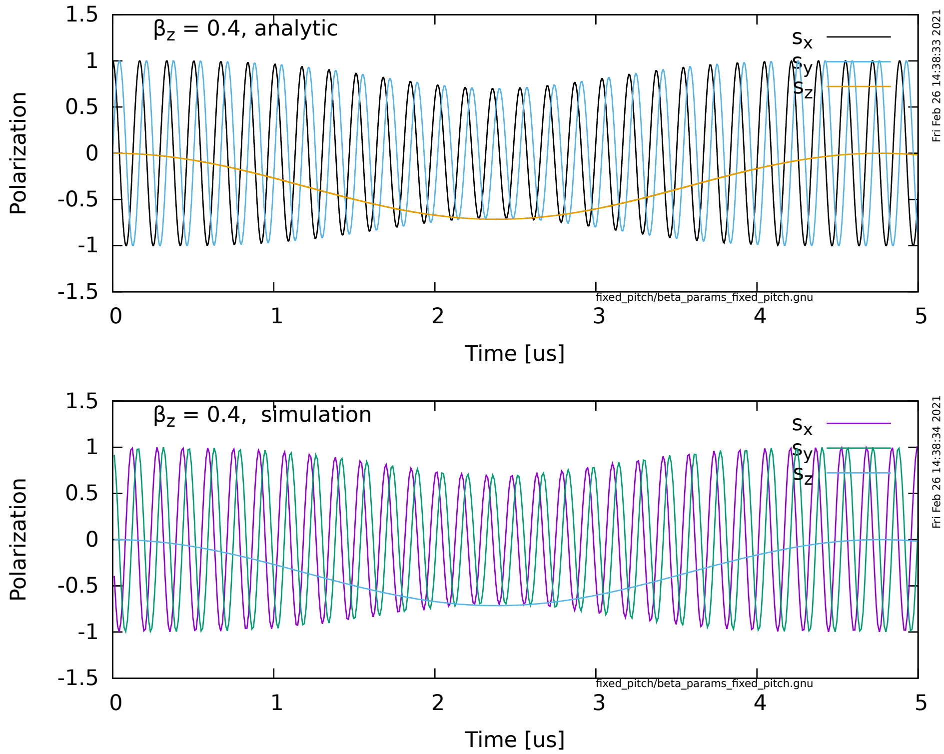


FIG. 5: Polarization of a muon following a spiraling trajectory, $\beta_z = 0.4$. The analytic solution (Equations 7) is on the top, and simulation (integration of the equations of motion and the BMT equation) at bottom.

direction of motion

$$\frac{d}{dt}(\boldsymbol{\beta}\cdot\mathbf{s}) = -\frac{\omega'}{2}\left[\left(1+\frac{1}{\omega'^2}[(\omega-\omega_0)^2-\Omega^2]\right)\beta_{xy}-\beta_z\frac{(\omega-\omega_0)\Omega}{\omega'^2}\right]\sin(\omega' t)$$

In general $\hat{\boldsymbol{\beta}}\cdot\mathbf{s}$ oscillates with frequency $\omega'$, the eigenvalue of the Hamiltonian in the rotating frame.

Finally, we use 9,10,11 to evaluate $\omega'$

$$\omega'^2 = (\omega_0-\omega)^2+\Omega^2 = \left(\frac{eB}{mc}\right)^2\left[\left(\frac{1}{\gamma}+a_\mu(\frac{1+\gamma(1-\beta_z^2)}{\gamma+1})-\frac{1}{\gamma}\right)^2+\left(a_\mu\frac{\gamma}{\gamma+1}\beta_z(\beta^2-\beta_z^2)^{1/2}\right)^2\right] \tag{14}$$

$$= \omega_a^2\left(\frac{1+\gamma(1-\beta_z^2))^2+\gamma^2\beta_z^2(\beta^2-\beta_z^2)}{(\gamma+1)^2}\right) = \omega_a^2\left(\frac{(1+2\gamma(1-\beta_z^2)+\gamma^2(1-\beta_z^2)^2+\gamma^2\beta_z^2(\beta^2-\beta_z^2)}{(\gamma+1)^2}\right)$$

$$= \omega_a^2\left(\frac{1+2\gamma(1-\beta_z^2)+\gamma^2(1-2\beta_z^2)+\gamma^2\beta_z^2\beta^2}{(\gamma+1)^2}\right)$$

$$= \omega_a^2\left(\frac{(1+2\gamma(1-\beta_z^2)+\gamma^2(1-2\beta_z^2))+\beta_z^2(\gamma^2-1)}{(\gamma+1)^2}\right)$$

$$= \omega_a^2\left(\frac{(1+2\gamma(1-\beta_z^2)+\gamma^2(1-\beta_z^2))-\beta_z^2}{(\gamma+1)^2}\right)$$

$$\omega' = \omega_a\left(\frac{(1+\gamma)(1-\beta_z^2)^{1/2}}{\gamma+1}\right)$$

$$\omega' = \omega_a(1-\beta_z^2)^{1/2} \tag{15}$$

$\omega'$ is the precession frequency measured in the rotating frame, the familiar pitch correction.

It turns out that the spiral trajectory corresponding to a fixed pitch can be transformed by a Lorentz boost ($\beta_z$) into a circular trajectory with zero pitch. Frequencies measured in the two frames are related by the time dilation factor $\gamma = \sqrt{1-\beta_z^2}$. The result is of course equivalent to 15. See XII D for details.

For $\beta_z \ll \beta$, and $\beta \sim 1$, $\beta_z = \psi$ and $\omega' = \omega_a(1-\frac{1}{2}\psi^2)$ where $\psi$ is the pitch angle, in good agreement with Farley[3], Silenko[4], Kim[5], Miller[6] and others.

### B. Equations of motion in a uniform magnetic field and vertically focusing electric field.

The next step is to include the electric field and vertical oscillations. In order to construct the effective magnetic field in our Hamiltonian we need to solve the equations of motion to determine the velocity. For motion in electromagnetic fields

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma mc}[\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})] \tag{16}$$

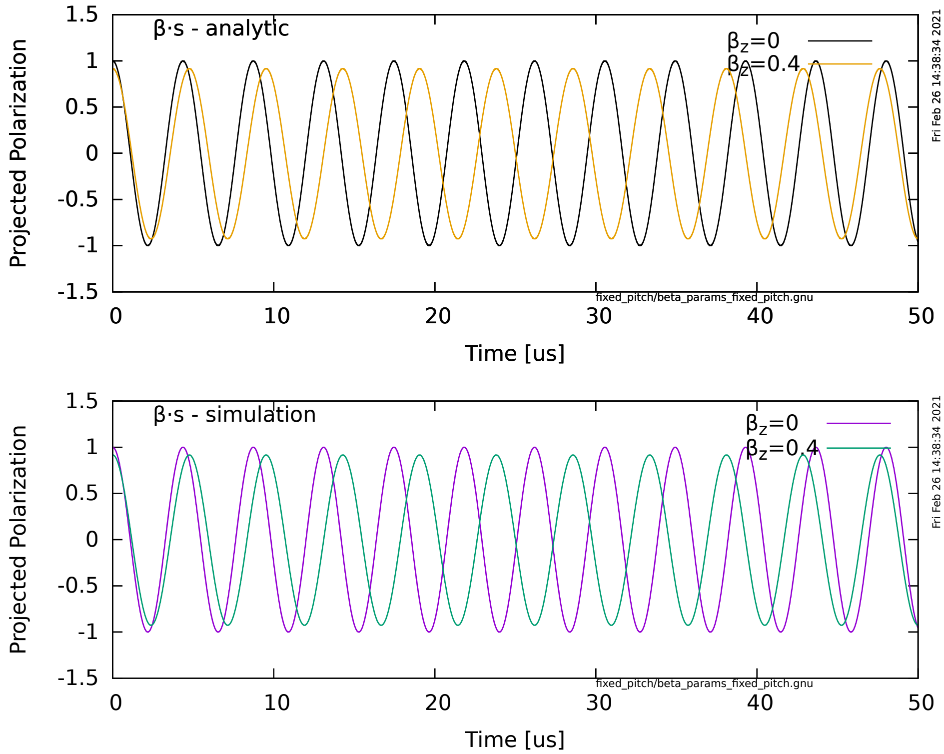


FIG. 6: Projection of the polarization onto the direction of motion for a muon following circular ($\beta_z = 0$ and spiraling $\beta_z = 0.4$ trajctories. The analytic solution (Equation 13) is on the top, and simulation (integration of the equations of motion and the BMT equation) at the bottom.

The electric and magnetic fields are both vertical (z-direction). With that stipulation 16 reduces to the coupled differential equations

$$\frac{d\beta_z}{dt} = \frac{e}{\gamma mc} E_z(1 - \beta_z^2) \tag{17}$$

$$\frac{d\beta_x}{dt} = \frac{e}{\gamma mc}(\beta_y B_z - \beta_x \beta_z E_z) \tag{18}$$

$$\frac{d\beta_y}{dt} = -\frac{e}{\gamma mc}(\beta_x B_z + \beta_y \beta_z E_z) \tag{19}$$

The tranverse momentum (momentum in the x-y plane) is conserved so that

$$\frac{p_{xy}}{m} = \gamma(t)\boldsymbol{\beta}_{xy}(t) = \gamma^0 \beta_{xy}^0 = \text{constant}$$

(Forces in the transverse plane are always perpendicular to the velocity.) The total momentum depends on the z-component of velocity.

$$
\begin{aligned}
\beta_{xy} = \beta_{xy}^{0}\frac{\gamma_0}{\gamma} &= \left[\frac{1-(\beta_{xy}^2+\beta_z^2)}{1-{\beta_{xy}^{0}}^{2}}\right]^{1/2} \\
\rightarrow \beta_{xy}^2(1-{\beta_{xy}^{0}}^{2}) &= {\beta_{xy}^{0}}^{2}(1-(\beta_{xy}^2+\beta_z^2)) \\
\beta_{xy}^2 &= {\beta_{xy}^{0}}^{2}(1-\beta_z^2) \\
\rightarrow \frac{\beta_{xy}}{\beta_{xy}^{0}} &= (1-\beta_z^2)^{1/2} = \frac{\gamma_0}{\gamma}
\end{aligned} \tag{20}
$$

and in an obvious notation, $\gamma = \gamma_z\gamma_0$. With that handy relationship we rewrite the first of the three coupled equations as

$$
\begin{aligned}
\frac{d\beta_z}{dt} &= \frac{e}{mc}E_z\frac{(1-\beta_z^2)}{\gamma_0\gamma_z} \\
&= \frac{e}{mc}E_z\frac{(1-\beta_z^2)^{3/2}}{\gamma_0}
\end{aligned} \tag{21}
$$

In the limit where $\beta_z \ll 1$, and assuming $E_z = -kz$ (vertically focusing linear restoring force and continuous quads)

$$
\begin{aligned}
\frac{d\beta_z}{dt} &\sim \frac{e}{\gamma_0 mc}E_z \\
\rightarrow \frac{1}{c}\frac{d^2z}{dt^2} &\sim -\frac{e}{\gamma_0 mc}kz \\
\rightarrow z = z_0\cos\omega_v t,\ \beta_z &= -\frac{z_0}{c}\omega_v\sin\omega_v t
\end{aligned}
$$

where

$$
\omega_v = \sqrt{\frac{ek}{\gamma_0 m}}
$$

We compute the next higher order term by substitution of the zeroth order solution into 21 and integrating. To order $\beta_z^2$,

$$\begin{aligned}
\beta_z &= -\frac{ekz_0}{\gamma_0 mc}\int \cos\omega_v t\left(1-\left(\frac{z_0}{c}\omega_v\right)^2\sin^2\omega_v t\right)^{3/2}dt \\
\beta_z &\sim -\frac{ekz_0}{\gamma_0 mc}\int \cos\omega_v t\left(1-\frac{3}{2}\left(\frac{z_0}{c}\omega_v\right)^2\sin^2\omega_v t\right)dt \\
&\sim -\frac{ekz_0}{\gamma_0 mc}\frac{1}{\omega_v}\left(\sin\omega_v t-\frac{1}{2}\left(\frac{z_0}{c}\omega_v\right)^2\sin^3\omega_v t\right) \\
&\sim -\omega_v\frac{z_0}{c}\sin\omega_v t[\cos\left(\left(\frac{z_0}{c}\omega_v\right)\sin\omega_v t\right)] \\
&\sim -\beta_z^0\sin\omega_v t[\cos\left(\beta_z^0\sin\omega_v t\right)] \\
&\sim -[\sin(\beta_z^0\sin\omega_v t)][\cos\left(\beta_z^0\sin\omega_v t\right)] \\
&\sim -\frac{1}{2}\sin\left(2\beta_z^0\sin\omega_v t\right)] \sim -\sin(\beta_z^0\sin\omega_v t)
\end{aligned} \tag{22}$$

Using 20 we write

$$\beta_{xy} = (1-\beta_z^2)^{1/2}\beta_{xy}^0 \sim \beta_{xy}^0\cos(\beta_z^0\sin\omega_v t) \tag{23}$$

and to second order in the small parameter $\beta_z^0$,

$$\boldsymbol{\beta} = \beta_{xy}^0\cos(\beta_z^0\sin\omega_v t)(\cos(\omega t)\hat{\mathbf{x}}-\sin(\omega t)\hat{\mathbf{y}})-\sin(\beta_z^0\sin\omega_v t)\hat{\mathbf{z}} \tag{24}$$

which is confirmed by substitution into Equations 17,18 and 19. See section XII C for details.

### C. Hamiltonian for uniform magnetic field and vertically focusing electric field

The electric field appears in the BMT equation in the cross product with velocity.

$$\begin{aligned}
\boldsymbol{\beta}\times\mathbf{E} &= \beta_{xy}(t)(-\sin(\omega t)\hat{\mathbf{x}}-\cos(\omega t))\hat{\mathbf{y}})E_z(t) \\
&= \beta_{xy}^0\gamma_z^{-1}(t)(\sin(\omega t)\hat{\mathbf{x}}+\cos(\omega t))\hat{\mathbf{y}})kz_0(\cos\omega_v t)
\end{aligned}$$

Including the electric field our effective magnetic field $\mathcal{B}$ becomes

$$\begin{aligned}
\mathcal{B} &= \Bigg[(a_\mu+\frac{1}{\gamma})B_z\hat{\mathbf{z}}-a_\mu\frac{\gamma}{\gamma+1}(\beta_z B_z)\left(\beta_z\hat{\mathbf{z}}+\beta_{xy}(t)(\cos\omega t\hat{\mathbf{x}}-\sin\omega t\hat{\mathbf{y}})\right) \\
&\quad -\left(a_\mu+\frac{1}{\gamma+1}\right)\beta_{xy}(t)E_z(t)(-\sin(\omega t)\hat{\mathbf{x}}-\cos(\omega t))\hat{\mathbf{y}})\Bigg] \\
&= B_z\left[\left((a_\mu+\frac{1}{\gamma})-a_\mu\frac{\gamma}{\gamma+1}\beta_z^2\right)\hat{\mathbf{z}}-\left(a_\mu\frac{\gamma}{\gamma+1}\beta_z\beta_{xy}(\cos(\omega t)\hat{\mathbf{x}}-\sin(\omega t)\hat{\mathbf{y}}\right)\right] \\
&\quad -\left(a_\mu+\frac{1}{\gamma+1}\right)\beta_{xy}(t)E_z(t)(-\sin(\omega t)\hat{\mathbf{x}}-\cos(\omega t))\hat{\mathbf{y}})
\end{aligned}$$

Then

$$\begin{aligned}\mathcal{H} &= -\tilde{\boldsymbol{\mu}} \cdot \mathcal{B} \\ &= -\frac{\hbar}{2}\begin{pmatrix} \omega_0(t) & (\Omega(t)+i\Pi(t))e^{i\omega t} \\ (\Omega(t)-i\Pi(t))e^{-i\omega t} & -\omega_0(t)\end{pmatrix}\end{aligned} \tag{25}$$

where

$$\Pi(t) = -\frac{e}{mc}\left(a_\mu + \frac{1}{\gamma+1}\right)E_z(t)\beta_{xy}(t) \tag{26}$$

and generalizing 9 and 10

$$\omega_0(t) \equiv \frac{e}{mc}B_z\left[a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\beta_z(t)^2\right] \tag{27}$$

$$\Omega(t) = -\frac{e}{mc}\left(B_z a_\mu \frac{\gamma}{\gamma+1}\beta_z(t)\beta_{xy}(t)\right) \tag{28}$$

Substituting the explicit time dependence of $\beta_z(t)$ and $E_z(t)$ into 27 and 26

$$\Omega(t) + i\Pi(t) = -z_0\frac{eB}{mc}\left(a_\mu\frac{\gamma}{\gamma+1}\frac{-\omega_v\sin\omega_v t}{c} + i\left(a_\mu + \frac{1}{\gamma+1}\right)\frac{\omega_v^2}{c\omega}\cos\omega_v t\right)\beta_{xy}(t) \tag{29}$$

where since $\omega_v^2 = \frac{ek}{\gamma m}$ and $\omega = \frac{eB}{\gamma mc}$, we can write $k = B\frac{\omega_v^2}{c\omega}$. Also from 28

$$\begin{aligned}\omega_0(t) &= \frac{eB}{mc}\left(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\beta_z^2\right) \\ &= \frac{eB}{mc}\left(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\left(\frac{z_0\omega_v\sin(\omega_v t)}{c}\right)^2\right) \\ &= \frac{eB}{mc}\left(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\left(\beta_z^0\sin(\omega_v t)\right)^2\right)\end{aligned} \tag{30}$$

$$\mathcal{H} = -\boldsymbol{\mu}\cdot\mathcal{B} = -\frac{e}{mc}\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\mathcal{B} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0(t) & (\Omega(t)+i\Pi(t))e^{i\omega t} \\ (\Omega(t)-i\Pi(t))e^{-i\omega t} & -\omega_0\end{pmatrix}$$

Having constructed the Hamiltonian, all that remains is to solve Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\psi = \mathcal{H}\psi \tag{31}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b\end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \omega_0(t) & (\Omega(t)+i\Pi(t))e^{i\omega t} \\ (\Omega(t)-i\Pi(t))e^{-i\omega t} & -\omega_0(t)\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} \tag{32}$$

### D. Time Dependent Perturbation Theory

We solve Schrodinger's equation perturbatively to second order in the small parameter $\beta_z^0 = -z_0\omega_v/c$. The reader is likely most familiar with using time dependent perturbation theory to compute transition rates between eigenstates of the unperturbed Hamiltonian. Our purpose is to determine a frequency shift in the observable $\boldsymbol{\beta}\cdot\mathbf{s}$. To see how we will extract a frequency shift from the Dyson series, we will study a problem for which we know the exact solution, namely, the Hamiltonian of Equation 12. If we treat the off diagonal element as a perturbation, which is valid if the pitch angle is small, then we can compute the frequency shift due to the perturbation and compare to the exact result. And we do just that in Appendix XII F.

### E. Second Order Perturbative Solution to Schrodinger equation for uniform magnetic field and vertically focusing electric field

Our goal is to solve 32 to second order in $\beta_z^0$. (It is convenient to adjust the phase of the vertical oscillation so that when $\omega_v \to 0$, the Hamiltonian takes the form of the system with fixed pitch, and to that end include a phase shift $\phi = \pi/2$.)

$$\begin{aligned}
\Omega(t) + i\Pi(t) &= -\Omega_0\sin(\omega_v t + \pi/2) + i\Pi_0\cos(\omega_v t + \pi/2) \\
&= -\Omega_0\cos(\omega_v t) - i\Pi_0\sin(\omega_v t) \\
&= \frac{1}{2}\left(-\Omega_0(e^{i\omega_v t} + e^{-i\omega_v t}) - \Pi_0(e^{i\omega_v t} - e^{-i\omega_v t})\right) \\
&= \frac{-1}{2}\left((\Omega_0 + \Pi_0)e^{i\omega_v t} + (\Omega_0 - \Pi_0)e^{-i\omega_v t})\right) \\
&= \left(X_0 e^{i\omega_v t} + Y_0 e^{-i\omega_v t})\right)
\end{aligned}$$

where

$$\begin{aligned}
\Omega_0 &= -z_0\frac{eB}{mc}\left(a_\mu\frac{\gamma}{\gamma+1}\frac{\omega_v}{c}\right)\beta_{xy}(t) \\
\Pi_0 &= -z_0\frac{eB}{mc}\left(a_\mu + \frac{1}{\gamma+1}\right)\frac{\omega_v^2}{c\omega}\beta_{xy}(t)
\end{aligned}$$

At the magic momentum, and with the substitution $\beta_z^0 = -z_0\omega_v/c$

$$X_0 = -\frac{\Omega_0 + \Pi_0}{2} = -\frac{1}{2}\beta_z^0\omega_a(\frac{\gamma}{\gamma+1} + \gamma\frac{\omega_v}{\omega})\beta_{xy} \tag{33}$$

$$Y_0 = -\frac{\Omega_0 - \Pi_0}{2} = -\frac{1}{2}\beta_z^0\omega_a(\frac{\gamma}{\gamma+1} - \gamma\frac{\omega_v}{\omega})\beta_{xy} \tag{34}$$

$$\tag{35}$$

Also

$$\omega_0(t) = \omega_0 - \pi_0\sin^2(\omega_v t + \pi/2) = \omega_0 - \frac{1}{2}\pi_0(1+\cos 2\omega_v t) = (\omega_0 - \frac{1}{2}\pi_0) - \frac{1}{2}\pi_0\cos 2\omega_v t$$
$$\equiv \omega_0' - \pi_0'\cos 2\omega_v t$$

with

$$\omega_0' = \omega_0 - \frac{\pi_0}{2} = \frac{eB}{mc}\left(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\frac{(\beta_z^0)^2}{2}\right) \quad \text{and} \quad \pi_0' = \frac{eB}{mc}\left(a_\mu\frac{\gamma}{\gamma+1}\frac{(\beta_z^0)^2}{2}\right)$$

and

$$\eta = (\omega_0' - \omega) = \omega_a(1 - \frac{\gamma}{\gamma+1}\frac{(\beta_z^0)^2}{2})$$

Then we can write,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$$

where in the rotating frame

$$\mathcal{H}_0 = -\frac{\hbar}{2}\begin{pmatrix}\omega_0' - \omega & 0 \\ 0 & -\omega_0' + \omega\end{pmatrix} = \hbar\begin{pmatrix}\eta/2 & 0 \\ 0 & -\eta/2\end{pmatrix}$$

$$\mathcal{H}' = \begin{pmatrix}H'_{aa} & H'_{ab} \\ H'_{ba} & H'_{bb}\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}\pi_0'\cos 2\omega_v t & X_0 e^{i\omega_v t} - Y_0 e^{-i\omega_v t} \\ X_0^* e^{-i\omega_v t} - Y_0^* e^{i\omega_v t} & -\pi_0'\cos 2\omega_v t\end{pmatrix}$$

The solutions to the unperturbed hamiltonian, ($\mathcal{H}_0$) are

$$a(t) = a_0 e^{-i\eta t/2}$$
$$b(t) = b_0 e^{i\eta t/2}$$

The general solution to the Schrodinger equation for the full Hamiltonian is written

$$\psi(t) = c_a(t)\psi_a e^{-i\eta t/2} + c_b(t)\psi_b e^{i\eta t/2} \tag{36}$$

$\psi_a$ and $\psi_b$ are a pair of eigenkets of $\mathcal{H}_0$. Substitution into the Schrodinger equation and some rearrangement gives us

$$\dot{c}_a = \frac{-i}{\hbar}[c_a H'_{aa} + c_b H'_{ab} e^{i\eta t}]$$
$$\dot{c}_b = \frac{-i}{\hbar}[c_b H'_{bb} + c_a H'_{ba} e^{i\eta t}] \tag{37}$$

Note that all of the terms in the perturbative part of the Hamiltonian are proportional to pitch angle, $\beta_z^0$ or $\beta_z^{0^2}$. We cannot solve the differential equations exactly, but for small pitch angle, the perturbation is small, and we solve by successive approximations. At $t = 0$, the muon is in the state $c_a(0), c_b(0)$ with polarization

$$\begin{aligned}
\langle \mathbf{s} \rangle &= \begin{pmatrix} c_a & c_b \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} c_a \\ c_b \end{pmatrix} \\
&= \begin{pmatrix} c_a & c_b \end{pmatrix} \sigma_x \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} c_a & c_b \end{pmatrix} \sigma_y \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{y}} + \begin{pmatrix} c_a & c_b \end{pmatrix} \sigma_z \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{z}} \\
&= \begin{pmatrix} c_a & c_b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} c_a & c_b \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{y}} + \begin{pmatrix} c_a & c_b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_a \\ c_b \end{pmatrix} \hat{\mathbf{z}}
\end{aligned}$$

If we choose $c_a(0) = c_b(0) = \frac{1}{\sqrt{2}}$ then the polarization $\mathbf{s}(0) = s_x$ is initially in the x-direction and the zeroth order solution is $c_a^{(0)} = c_b^{(0)} = \frac{1}{\sqrt{2}}$. To compute the first order approximation substitute the zeroth order solution into 37 and integrate

$$c_a^{(1)} = \frac{-i}{\hbar}\int_0^t [c_a^{(0)} H'_{aa} + c_b^{(0)} H'_{ab} e^{i\eta t'}]dt'$$
$$c_b^{(1)} = \frac{-i}{\hbar}\int_0^t [c_b^{(0)} H'_{bb} + c_a^{(0)} H'_{ba} e^{-i\eta t'}]dt'$$

We know that in the case of a fixed pitch angle ($\beta_z$ constant), the precession frequency depends on $\beta_z^2$, so we will need to go at least to second order. But now that is straightforward.

$$c_a^{(2)} = \frac{-i}{\hbar}\int_0^t [c_a^{(1)} H'_{aa} + c_b^{(1)} H'_{ab} e^{i\eta t'}]dt'$$
$$c_b^{(2)} = \frac{-i}{\hbar}\int_0^t [c_b^{(1)} H'_{bb} + c_a^{(1)} H'_{ba} e^{-i\eta t'}]dt'$$

Then we will have

$$c_a(t) = c_a^{(0)} + c_a^{(1)} + c_a^{(2)} + \ldots$$
$$c_b(t) = c_b^{(0)} + c_b^{(1)} + c_b^{(2)} + \ldots$$

In order to simplify the calculation it is helpful to take advantage of the known symmetries of $\mathcal{H}'$, in particular $H'_{bb} = -H'_{aa}$ and $H'_{ba} = {H'_{ab}}^*$, and introduce a bit of notation. Define

$$A_1 = \frac{1}{\hbar}\int_0^t [H'_{aa}]dt'$$
$$B_1 = \frac{1}{\hbar}\int_0^t [H'_{ab}e^{i\eta t'}]dt'$$

Then

$$c_a^{(1)} = -i(c_a^{(0)}A_1 + c_b^{(0)}B_1 \to \frac{-i}{\sqrt{2}}(A_1 + B_1)$$
$$c_b^{(1)} = i(c_a^{(0)}A_1 - c_v^{(0)}B_1^*) \to \frac{i}{\sqrt{2}}(A_1 - B_1)$$

where we set $c_a^{(0)} = c_b^{(0)} = \frac{1}{\sqrt{2}}$. Next since

$$c_a^{(2)} = \frac{-i}{\hbar}\frac{1}{\sqrt{2}}\int_0^t [-i(A_1 + B_1)H'_{aa} - i(-A_1 + B_1^*)H'_{ab}e^{i\eta t'}]dt'$$
$$c_b^{(2)} = \frac{-i}{\hbar}\frac{1}{\sqrt{2}}\int_0^t [-i(-A_1 + B_1^*)(-H'_{aa}) - i(A_1 + B_1)(H'_{ab}e^{i\eta t'})^*]dt'$$

it is convenient to also define

$$C_2 = \frac{1}{\hbar}\int_0^t [A_1 H'_{aa}]dt'$$
$$D_2 = \frac{1}{\hbar}\int_0^t [B_1 H'_{aa}]dt'$$
$$E_2 = \frac{1}{\hbar}\int_0^t [(-A_1)H'_{ab}e^{i\eta t'}]dt'$$
$$F_2 = \frac{1}{\hbar}\int_0^t [B_1^* H'_{ab}e^{i\eta t'}]dt'$$

so that

$$c_a^{(2)} = -\frac{1}{\sqrt{2}}(C_2 + D_2 + E_2 + F_2)$$
$$c_b^{(2)} = -\frac{1}{\sqrt{2}}(C_2 - D_2^* - E_2^* + F_2^*)$$

In that last step we use the fact that $H'_{aa}$ is real, and with the initial condition that $c_a(0) = c_b(0) = \frac{1}{\sqrt{2}}$, $A_1$ is also real. Then

$$\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} c_a(t)e^{-i\eta t/2} \\ c_b(t)e^{i\eta t/2} \end{pmatrix} \tag{38}$$

The observable is the projection of the polarization onto the velocity.

$$\boldsymbol{\sigma}\cdot\boldsymbol{\beta} = \begin{pmatrix} \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & -\beta_z \end{pmatrix} = \begin{pmatrix} -\beta_z^0 \sin\omega_v t & -\beta_{xy}^0 \cos(\beta_z^0 \sin\omega_v t)e^{-i\omega t} \\ -\beta_{xy}\cos(\beta_z^0 \sin\omega_v t)e^{i\omega t} & \beta_z^0 \sin\omega_v t \end{pmatrix}$$

with expectation value

$$\begin{aligned} &= -(|a|^2 - |b|^2)\beta_z^0 \sin(\omega_v t) - (a^* b)\beta_{xy}^0 \cos(\beta_z^0 \sin\omega_v t)e^{-i\omega t} - (b^* a)\beta_{xy}\cos(\beta_z^0 \sin\omega_v t)e^{i\omega t} \\ &= -(|a|^2 - |b|^2)\beta_z^0 \sin(\omega_v t) - 2\Re\left((a^* b)\beta_{xy}^0 \cos(\beta_z^0 \sin\omega_v t)e^{-i\omega t}\right) \end{aligned}$$

Let's write out $a(t)$ and $b(t)$

$$\begin{aligned} a(t) &= \frac{1}{\sqrt{2}}\left(1 - i(A_1 + B_1) - (C_2 + D_2 + E_2 + F_2)\right)e^{-i(\eta+\omega)t/2} \\ a^*(t) &= \frac{1}{\sqrt{2}}\left(1 + i(A_1 + B_1^*) - (C_2 + D_2^* + E_2^* + F_2^*)\right)e^{i(\eta+\omega)t/2} \\ b(t) &= \frac{1}{\sqrt{2}}\left(1 + i(A_1 - B_1^*) - (C_2 - D_2^* - E_2^* + F_2^*)\right)e^{i(\eta+\omega)t/2} \\ b^*(t) &= \frac{1}{\sqrt{2}}\left(1 - i(A_1 - B_1) - (C_2 - D_2 - E_2 + F_2)\right)e^{-i(\eta+\omega)t/2} \end{aligned}$$

where that factor $e^{i\omega t/2}$ transforms from rotating back to laboratory frame. The observables are linear combinations of $|a|^2 - |b|^2|$ and $a^* b$ To second order in the small parameter $\beta_z^0$

$$\begin{aligned} |a|^2 - |b|^2 &\sim \left(1 - i(B_1 - B_1^*) - (2C_2 + D_2 + D_2^* + (E_2 + E_2^*) + (F_2 - F_2^*) + |B_1|^2\right) \\ &\quad - \left(1 + i(B_1 - B_1^*) - (2C_2 - (D_2 + D_2^*) - (E_2 + E_2^*) + (F_2 - F_2^*) + |B_1|^2\right) \\ &\sim -2i\left((B_1 - B_1^*) - 2(D_2 + D_2^*) - 2(E_2 + E_2^*)\right) \\ &\sim -2i(B_1 - B_1^*) \end{aligned} \tag{39}$$

$$\begin{aligned} a^* b &\sim \frac{1}{2}\left(1 + 2iA_1 + (B_1^*)^2 - 2(C_2 + F_2^*)\right)e^{i(\eta+\omega)t} \\ &\sim \frac{1}{2}\left(1 + 2iA_1 + (B_1^*)^2 - 2F_2^*\right)e^{i(\eta+\omega)t} \end{aligned} \tag{40}$$

where in the end we dropped all terms that are order higer than ${\beta_z^0}^2$. (Note that $A_1 \sim \beta_z^2$, $B_1 \sim \beta_z$, $F_2 \sim \beta_z^2$ and $C_2, D_2, E_2 \sim \beta_z^3$). We need only to evaluate $A_1, B_1$ and $F_2$.

$$A_1 = \frac{1}{\hbar}\int_0^t H'_{aa}dt' = -\int_0^t \frac{\pi'_0}{2}\cos 2\omega_v t' dt' = -\frac{\pi'_0}{2}\frac{\sin 2\omega_v t}{2\omega_v} \tag{41}$$

$$\lim_{\omega_v \to 0} A_1(\omega_v) = -\frac{\pi'_0}{2}t \tag{42}$$

$$B_1 = \frac{1}{\hbar}\int_0^t H'_{ab}e^{i\eta t'}dt' = \int_0^t \frac{1}{2}(X_0 e^{i\omega_v t'} + Y_0 e^{-i\omega_v t'})e^{i\eta t'}dt' = \frac{1}{2}\left(X_0\frac{e^{i(\omega_v+\eta)t}-1}{i(\omega_v+\eta)} + Y_0\frac{e^{-i(\omega_v-\eta)t}-1}{-i(\omega_v-\eta)}\right) \tag{43}$$

$$B_1 - B_1^* = -i\left(X_0\frac{\cos(\omega_v+\eta)t-1}{\omega_v+\eta} - Y_0\frac{\cos(\omega_v-\eta)t-1}{\omega_v-\eta}\right) \tag{44}$$

$$\begin{aligned}
(B_1^*)^2 &= \frac{1}{4}\left(4(X_0^*)^2\frac{e^{-i(\omega_v+\eta)t}\sin^2(\omega_v+\eta)t/2}{(\omega_v+\eta)^2} + 2X_0^*Y_0^*\frac{e^{-2i\eta t}-2e^{-i\eta t}\cos(\omega_v t)+1}{(\omega_v^2-\eta^2)}\right.\\
&\left.+\ 4(Y_0^*)^2\frac{e^{i(\omega_v-\eta)t}\sin^2(\omega_v-\eta)t/2}{(\omega_v-\eta)^2}\right)\\
&= e^{-i\eta t}\left(X_0^2\frac{e^{-i(\omega_v)t}\sin^2(\omega_v+\eta)t/2}{(\omega_v+\eta)^2} + \frac{X_0Y_0}{2}\frac{e^{-i\eta t}-2\cos(\omega_v t)+e^{i\eta t}}{(\omega_v^2-\eta^2)} + Y_0^2\frac{e^{i(\omega_v)t}\sin^2(\omega_v-\eta)t/2}{(\omega_v-\eta)^2}\right)\\
&= e^{-i\eta t}\left(X_0^2\frac{e^{-i(\omega_v)t}\sin^2(\omega_v+\eta)t/2}{(\omega_v+\eta)^2} + X_0Y_0\frac{(\cos\eta t-\cos\omega_v t)}{(\omega_v^2-\eta^2)} + Y_0^2\frac{e^{i(\omega_v)t}\sin^2(\omega_v-\eta)t/2}{(\omega_v-\eta)^2}\right)
\end{aligned} \tag{45}$$

$$\begin{aligned}
F_2 &= \frac{1}{\hbar}\int_0^t B_1^* H'_{ab}e^{i\eta t'}dt' = \frac{1}{2}\int_0^t\left(X_0\frac{e^{-i(\omega_v+\eta)t'}-1}{-i(\omega_v+\eta)} + Y_0\frac{e^{i(\omega_v-\eta)t'}-1}{i(\omega_v-\eta)}\right)\frac{1}{2}(X_0e^{i\omega_v t'}+Y_0e^{-i\omega_v t'})e^{i\eta t'}dt'\\
&= \frac{1}{4}\left(\frac{1}{-i(\omega_v+\eta)}\left(X_0^2(t-\frac{e^{i(\omega_v+\eta)t}-1}{i(\omega_v+\eta)}) + X_0Y_0(\frac{e^{-2i\omega_v t}-1}{-2i\omega_v} - \frac{e^{-i(\omega_v-\eta)t}-1}{-i(\omega_v-\eta)})\right)\right.\\
&+\ \left.\frac{1}{i(\omega_v-\eta)}\left(Y_0X_0(\frac{e^{2i\omega_v t}-1}{2i\omega_v} - \frac{e^{i(\omega_v+\eta)t}-1}{i(\omega_v+\eta)}) + Y_0^2(t-\frac{e^{-i(\omega_v-\eta)t}-1}{-i(\omega_v-\eta)})\right)\right)\\
&= \frac{1}{4}\left(\frac{it[(\omega_v-\eta)X_0^2-(\omega_v+\eta)Y_0^2]}{\omega_v^2-\eta^2}\right.\\
&+\ X_0Y_0\left[\frac{e^{-i(\omega_v-\eta)t}-1+e^{i(\omega_v+\eta)t}-1}{\omega_v^2-\eta^2} + \frac{\sin\omega_v t}{\omega_v}(\frac{e^{-i\omega_v t}}{-i(\omega_v+\eta)} + \frac{e^{i\omega_v t}}{i(\omega_v-\eta})\right]\\
&-\ \left.X_0^2\frac{e^{i(\omega_v+\eta)t}-1}{(\omega_v+\eta)^2} - Y_0^2\frac{e^{-i(\omega_v-\eta)t}-1}{(\omega_v-\eta)^2}\right)
\end{aligned} \tag{46}$$

$$\begin{aligned}
&= \frac{1}{4}\left(\frac{it[(\omega_v(X_0^2-Y_0^2)-\eta(X_0^2+Y_0^2)]}{\omega_v^2-\eta^2} + 2X_0Y_0\left[\frac{e^{i\eta t}\cos\omega_v t-1}{\omega_v^2-\eta^2} + \frac{\sin\omega_v t}{\omega_v}\frac{(\omega_v\sin\omega_v t - i\eta\cos\omega_v t)}{(\omega_v^2-\eta^2)}\right]\right.\\
&-\ \left.X_0^2\frac{e^{i(\omega_v+\eta)t}-1}{(\omega_v+\eta)^2} - Y_0^2\frac{e^{-i(\omega_v-\eta)t}-1}{(\omega_v-\eta)^2}\right)\\
&= it\lambda/2 + F_{2-rem}
\end{aligned} \tag{47}$$

where we define

$$\lambda/2 \equiv \frac{1}{4}\left(\frac{[(\omega_v(X_0^2 - Y_0^2) - \eta(X_0^2 + Y_0^2)]}{\omega_v^2 - \eta^2}\right)$$

$$F_{2-rem} \equiv \frac{1}{4}\left(2X_0Y_0\left[\frac{e^{i\eta t}\cos\omega_v t - 1}{\omega_v^2 - \eta^2} + \frac{\sin\omega_v t}{\omega_v}\frac{(\omega_v \sin\omega_v t - i\eta\cos\omega_v t)}{(\omega_v^2 - \eta^2)}\right] + X_0^2\frac{e^{i(\omega_v+\eta)t} - 1}{(\omega_v + \eta)^2} - Y_0^2\frac{e^{-i(\omega_v-\eta)t} - }{(\omega_v - \eta)^2}\right.$$

For the special case where the $\omega_v = 0$, (zero electric field and fixed pitch angle)

$$\begin{aligned}
\lim_{\omega_v \to 0} F_2 &= \frac{1}{4}\left(\frac{-it(\eta(X_0^2 + Y_0^2)}{-\eta^2} + 2X_0Y_0\left[\frac{e^{i\eta t} - 1}{-\eta^2} + \frac{i\eta t}{\eta^2}\right] - X_0^2\frac{e^{i\eta t} - 1}{\eta^2} - Y_0^2\frac{e^{i\eta t} - 1}{\eta^2}\right) \\
&= \frac{X_0^2}{2\eta^2}\left(i\eta t + \left[-(e^{i\eta t} - 1) + i\eta t\right] - (e^{i\eta t} - 1)\right) \\
&= \frac{X_0^2}{\eta^2}\left(i\eta t - (e^{i\eta t} - 1)\right) = i\lambda'/2t - \frac{X_0^2}{\eta^2}(e^{i\eta t} - 1) \qquad (48)
\end{aligned}$$

(Note that for $\omega_v = 0$, $X_0 = Y_0$.)

Then we can rewrite 40 as

$$\begin{aligned}
a^*b &\sim \frac{1}{2}(1 - it\lambda + 2iA_1 + (B_1^*)^2 - 2F_{2-rem}^*)e^{i(\eta+\omega)t} \\
&\sim \frac{1}{2}(1 + 2iA_1 + (B_1^*)^2 - 2F_{2-rem}^*)e^{i(\eta+\omega-\lambda)t} \qquad (49)
\end{aligned}$$

and in the $\omega_v \to 0$ limit, when $A_1 \to -\pi_0'/2$ (see 42), and $\lambda \to \lambda'$ (see 48)

$$a^*b \sim \frac{1}{2}(1 + (B_1^*)^2 - 2F_{2-rem}'^*)e^{i(\eta+\omega+\lambda'-\pi_0')t} \qquad (50)$$

(To get from 43 to 44 we use the fact that $X_0$ and $Y_0$ are real.)

We learned in Appendix XII F that the second order frequency shift appears in the observables as the coefficients of $it$ (imaginary time). There is no such term in 39. However, $F_2^*$ does contain such a term, $\lambda$ that contributes to a second order frequency shift in $a^*b$. Recall that $a^*b$ corresponds to the polarization in the x-y plane. There is an additional contribution to the frequency shift, $\pi_0'$ in the $\omega_v = 0$ limit from the $A_1$ term.

The precession frequency in the rotating frame is $\omega'$.

if $\omega_v > 0$, then

$$\omega' \sim \eta + \omega - \lambda = \omega_0' - \frac{\pi_0}{2} + \lambda. \qquad (51)$$

if $\omega_v = 0$, then

$$\omega' \sim \eta + \omega - \lambda = \omega_0' - \frac{\pi_0}{2} + \lambda' - \pi_0' = \omega_0' - \pi_0 + 2\lambda. \qquad (52)$$

There is a discontinuity in the precession frequency at $\omega_v = 0$. If $\omega_v = 0$ there is no oscillation, the pitch angle is fixed, and $\langle\beta_z^2\rangle = (\beta_z^0)^2$. For $\omega_v > 0$,$\langle\beta_z^2\rangle = \frac{1}{2}(\beta_z^0)^2$.

At the magic momentum

$$X_0^2 - Y_0^2 = (\beta_z^0)^2\omega_a^2\frac{\gamma^2}{\gamma+1}\frac{\omega_v}{\omega}\beta_{xy}^2$$
$$X_0^2 + Y_0^2 = \frac{1}{2}(\beta_z^0)^2\omega_a^2\left(\frac{\gamma^2}{(\gamma+1)^2} + \gamma^2\frac{\omega_v^2}{\omega^2}\right)\beta_{xy}^2$$

and

$$\lambda = \frac{1}{4}(\beta_z^0)^2\omega_a^2\beta_{xy}^2\left(\frac{2\omega_v(\frac{\gamma^2}{\gamma+1}\frac{\omega_v}{\omega}) - \eta(\frac{\gamma^2}{(\gamma+1)^2} + \gamma^2\frac{\omega_v^2}{\omega^2})}{\omega_v^2 - \eta^2}\right)$$

and since $\eta = \omega_a + \mathcal{O}(\beta_z^0)^2$ we might as well replace $\eta$ with $\omega_a$ on the right hand size

$$\lambda = \frac{1}{4}(\beta_z^0)^2\omega_a^2\beta_{xy}^2\left(\frac{2\omega_v(\frac{\gamma^2}{\gamma+1}\frac{\omega_v}{\omega}) - \omega_a(\frac{\gamma^2}{(\gamma+1)^2} + \gamma^2\frac{\omega_v^2}{\omega^2})}{\omega_v^2 - \omega_a^2}\right) \tag{53}$$

If $\omega_v = 0$, then

$$\omega' = \omega_a(1 - (\beta_z^0)^2\left(\frac{\gamma}{\gamma+1} - \frac{1}{2}\frac{\gamma^2}{(\gamma+1)^2}\beta_{xy}^2\right))$$
$$\sim \omega_a(1 - \frac{1}{2}(\beta_z^0)^2)$$

If $\omega_v > 0$

$$\omega' = \omega_a\left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{1}{2}\omega_a\beta_{xy}^2\left(\frac{2\omega_v(\frac{\gamma^2}{\gamma+1}\frac{\omega_v}{\omega}) - \omega_a(\frac{\gamma^2}{(\gamma+1)^2} + \gamma^2\frac{\omega_v^2}{\omega^2})}{\omega_v^2 - \omega_a^2}\right)\right]\right) \tag{54}$$

In the limit where $\omega_v \gg \omega_a$

$$\begin{aligned}
\omega' &= \omega_a\left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{1}{2}\frac{\omega_a}{\omega}\beta_{xy}^2\left(\frac{2\gamma^2}{\gamma+1} - \gamma^2\frac{\omega_a}{\omega}\right)\right]\right)\\
&= \omega_a\left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{1}{2}\gamma a_\mu(\gamma^2-1)\left(\frac{2}{\gamma+1} - \gamma a_\mu\right)\right]\right)\\
&= \omega_a\left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{\gamma}{2}\left(\frac{2}{\gamma+1} - \frac{\gamma}{\gamma^2-1}\right)\right]\right)\\
&= \omega_a\left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{\gamma}{2}\left(\frac{\gamma-2}{\gamma^2-1}\right)\right]\right)\\
&= \omega_a\left(1 - \frac{(\beta_z^0)^2}{4}\left[\frac{\gamma^2}{(\gamma^2-1)}\right]\right)
\end{aligned}$$

The precession frequency $\omega'$ is plotted as a function of $\omega_v$ in Figure 7 for muons at the magic momentum and with pitch angle $\beta_z^0 = 0.01$. At $\omega_v \ll \omega'$, $\omega' = \omega_a(1-\frac{1}{4}(\beta_z^0)^2)$, whereas if $\omega_v \gg \omega'$, $\omega' = (\omega_a(1 - \frac{1}{4}(\beta_z^0)^2\frac{\gamma^2}{\gamma^2-1})$. In the g-2 experiment $\omega_v \sim 10\omega'$.

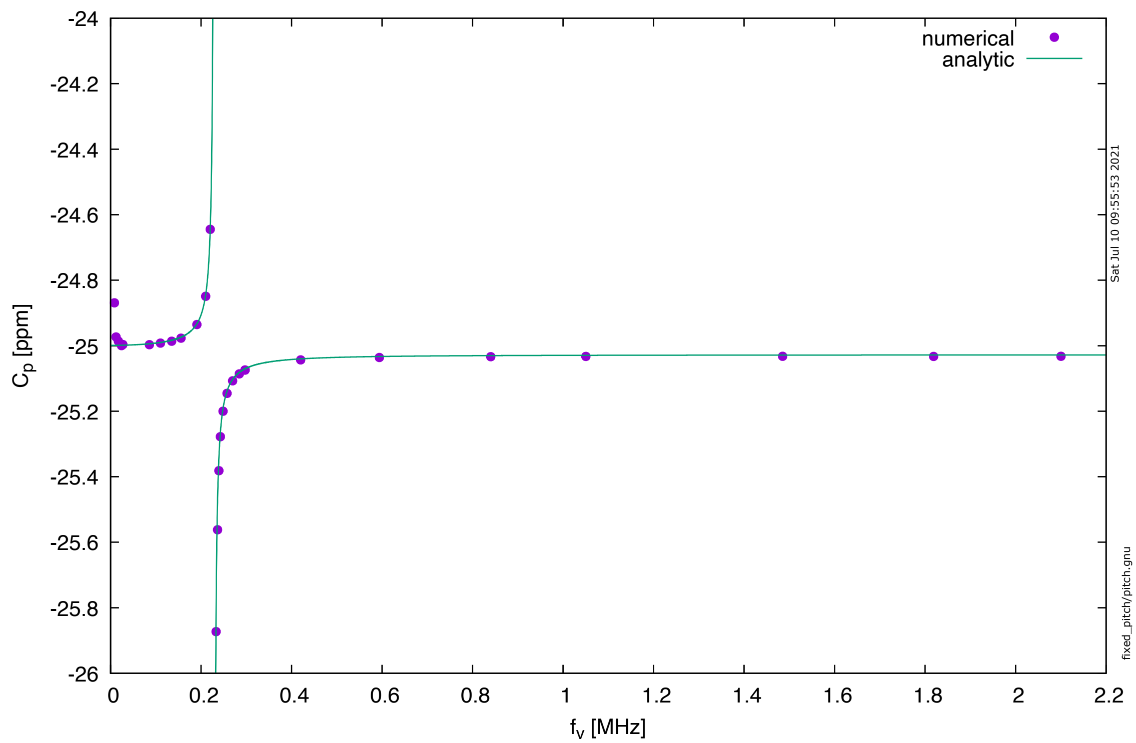


FIG. 7: The simulation (points) integrates the equations of motion and the Thomas-BMT equation to determine $\hat{\boldsymbol{\beta}} \cdot \mathbf{s}(t)$ and the frequency is extracted by a fit. The fitted frequency represents an average precession frequency over 2000 turns of the muon around the ring. The solid curve is Equation 54 assembled from Equations 51 with 53. When the betatron frequency $f_v$ is very low, there are an insufficient number of betatron periods to determine an average precession frequency, as evidenced by the discrepancy at low frequency in the plot. The nominal precession frequency, $\omega_a = \frac{eB}{mc} a_\mu = 2\pi(0.23\text{MHz})$.

## VII. RADIAL MAGNETIC FIELD

Suppose there is a uniform radial field (in addition to the uniform vertical magnetic field), and that the force on the muons due to the radial field is balanced by an equal and opposite force due to a vertical electric field, presumably the quadrupole field.

$$\mathbf{B} = B_r(\sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{y}}) + B_z\hat{\mathbf{z}}$$

$$\mathbf{E} = E_z\hat{\mathbf{z}}$$

The velocity is clockwise

$$\boldsymbol{\beta} = \beta(\cos\omega t\ \hat{\mathbf{x}} - \sin\omega t\ \hat{\mathbf{y}})$$

so that

$$\mathbf{B} = B_r(\sin\omega t\hat{\mathbf{x}} + \cos\omega t\hat{\mathbf{y}}) + B_z\hat{\mathbf{z}}$$

where $\theta = \omega t$, and

$$\boldsymbol{\beta}\times\mathbf{E} = E_z\beta(\sin\omega t\ \hat{\mathbf{x}} + \cos\omega t\ \hat{\mathbf{y}})$$

Note that $\boldsymbol{\beta}\times\mathbf{E}$ is in the radial direction.

$$\begin{aligned}
\mathcal{B} &= \left[(a_\mu + \frac{1}{\gamma})\mathbf{B} - (a_\mu + \frac{1}{\gamma+1})\beta\times\mathbf{E}\right] \\
\mathcal{H} &= -\tilde{\boldsymbol{\mu}}\cdot\mathcal{B} \\
&= -\frac{e\hbar}{mc}\frac{\boldsymbol{\sigma}}{2}\cdot\mathcal{B} \\
&= -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & (\Omega_r - \Pi_r)e^{i\omega t} \\ (\Omega_r - \Pi_r)e^{i\omega t} & -\omega_0 \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\omega_0 &= \frac{e}{mc}B_z(a_\mu + \frac{1}{\gamma}) \\
\omega &= \frac{e}{mc\gamma}B_z \\
\Omega_r &= -i\frac{e}{mc}B_r(a_\mu + \frac{1}{\gamma}) \\
\Pi_r &= -i\frac{e}{mc}E_z(a_\mu + \frac{1}{\gamma+1})\beta
\end{aligned}$$

By inspection

$$\begin{aligned}
\omega'^2 &= (\omega_0 - \omega)^2 + |\Omega_r - \Pi_r|^2 \\
&= \left(\frac{eB_z}{mc}\right)^2\left[a_\mu^2 + \left(\frac{B_r}{B_z}(a_\mu + \frac{1}{\gamma}) - \frac{E_z\beta}{B_z}(a_\mu + \frac{1}{\gamma+1})\right)^2\right]
\end{aligned}$$

As the vertical forces sum to zero, $E_z = \beta B_r$ and

$$\begin{aligned}
\omega'^2 &= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_r}{B_z}\right)^2 \left(a_\mu + \frac{1}{\gamma} - \beta^2(a_\mu + \frac{1}{\gamma+1})\right)^2\right] \\
&= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_r}{B_z}\right)^2 \left(a_\mu + \frac{1}{\gamma} - \frac{\gamma^2-1}{\gamma^2}(a_\mu + \frac{1}{\gamma+1})\right)^2\right] \\
&= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_r}{B_z}\right)^2 \left(\frac{1}{\gamma} + \frac{a_\mu}{\gamma^2} - \frac{\gamma-1}{\gamma^2}\right)^2\right] \\
&= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_r}{B_z}\right)^2 \left(\frac{a_\mu}{\gamma^2} + \frac{1}{\gamma^2}\right)^2\right] \\
&= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_r}{B_z}\right)^2 \left(\frac{a_\mu+1}{\gamma^2}\right)^2\right] \\
&= \left(\frac{eB_z}{mc}a_\mu\right)^2 \left[1 + \left(\frac{B_r}{B_z}\right)^2 \left(\frac{1+1/a_\mu}{\gamma^2}\right)^2\right]
\end{aligned}$$

At the magic momentum, $a_\mu = 1/(\gamma^2 - 1)$ and

$$\begin{aligned}
\omega' &= \left(\frac{eB_z}{mc}a_\mu\right) \left[1 + \left(\frac{B_r}{B_z}\right)^2 \left(\frac{1+\gamma^2-1}{\gamma^2}\right)^2\right]^{1/2} \\
\omega' &= \omega_a \left[1 + \left(\frac{B_r}{B_z}\right)^2\right]^{1/2}
\end{aligned}$$

as shown previously by Miller[7] by direct solution of the Thomas-BMT equation.

## VIII. LONGITUDINAL MAGNETIC FIELD

Consider the case where there is a longitudinal magnetic field, (in addition to the vertical field), $\mathbf{B} = B_l(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{y}}) + B_z\hat{\mathbf{z}}$. Since the velocity is everywhere parallel to the longitudinal field the trajectory of the muon is unaffected. The velocity

$$\boldsymbol{\beta} = \beta(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})$$

$$
\begin{aligned}
\mathcal{B} &= \left[(a_\mu + \frac{1}{\gamma})\mathbf{B} - a_\mu(\frac{\gamma}{\gamma+1})(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}\right] \\
&= \left[(a_\mu + \frac{1}{\gamma})(B_z\hat{\mathbf{z}} + B_l(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{y}})) - a_\mu(\frac{\gamma}{\gamma+1})\beta^2 B_l(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\right] \\
&= \left[(a_\mu + \frac{1}{\gamma})(B_z\hat{\mathbf{z}}) + B_l(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\beta^2)(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\right] \\
&= \left[(a_\mu + \frac{1}{\gamma})(B_z\hat{\mathbf{z}}) + B_l(\frac{1}{\gamma} + a_\mu(1 - \frac{\gamma}{\gamma+1}\beta^2))(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\right]
\end{aligned}
$$

Then

$$
\mathcal{H} = -\tilde{\boldsymbol{\mu}}\cdot\mathcal{B} = -\frac{e}{mc}\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega_l e^{i\omega t} \\ \Omega_l e^{-i\omega t} & \omega_0 \end{pmatrix}
$$

where

$$
\begin{aligned}
\omega_0 &= \frac{e}{mc}B_z(a_\mu + \frac{1}{\gamma}) \\
\omega &= \frac{e}{mc\gamma}B_z \\
\Omega_l &= \frac{e}{mc}B_l\left[\frac{1}{\gamma} + a_\mu(1 - \frac{\gamma}{\gamma+1}\beta^2)\right]
\end{aligned}
$$

### A. Uniform Longitudinal Field

In general the longitudinal field may depend on azimuthal angle. In the case that $B_l$ is independent of azimuth its affect on the precession frequency can be determined exactly. The eigenvalue in the rotating frame

$$
\begin{aligned}
{\omega'}^2 &= (\omega_0 - \omega)^2 + |\Omega_l|^2 \\
&= \left(\frac{eB_z}{mc}\right)^2\left[a_\mu^2 + \left(\frac{B_l}{B_z}\right)^2\left(\frac{1}{\gamma} + a_\mu(1 - \frac{\gamma\beta^2}{\gamma+1})\right)^2\right] \\
&= \left(\frac{eB_z}{mc}\right)^2\left[a_\mu^2 + \left(\frac{B_l}{B_z}\right)^2\left(\frac{1}{\gamma} + \frac{a_\mu}{\gamma}\right)^2\right]
\end{aligned}
$$

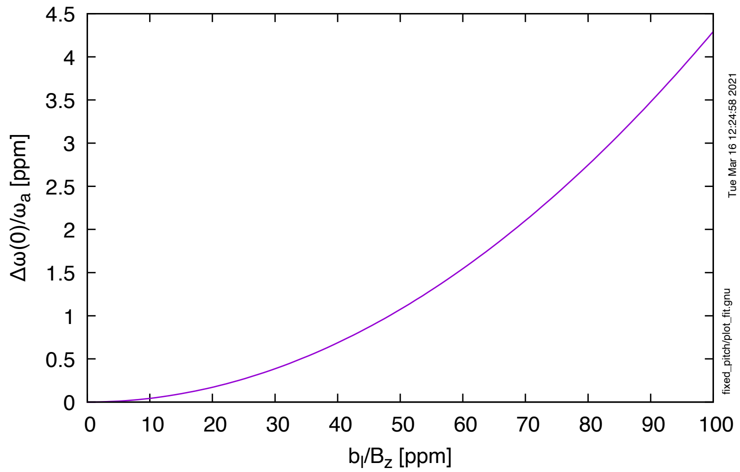


FIG. 8: Fractional shift in $\omega_a$ due to the ring average longitudinal field.

At the magic momentum

$$\begin{aligned} \omega'^2 &= \left(\frac{eB_z}{mc}\right)^2 \left[a_\mu^2 + \left(\frac{B_l}{B_z}\right)^2 (\gamma a_\mu)^2\right] \\ &= \omega_a^2 \left[1 + \left(\frac{B_l}{B_z}\right)^2 \gamma^2\right] \\ \rightarrow \frac{\Delta\omega'}{\omega_a} &\sim \frac{1}{2}\left(\frac{B_l}{B_z}\right)^2 \gamma^2 \end{aligned} \tag{55}$$

as shown previously by Miller[7] by direct solution of the Thomas-BMT equation. Note that we define $\omega_a = \frac{eB_z}{mc} a_\mu$, that is with respect to the vertical component of the magnetic field rather than the magnitude of the field $|B| = \sqrt{B_z^2 + B_l^2}$. The dependence of precession frequency in the rotating frame on longitudinal field is shown in Figure 8.

The polarization is given by (see section XII B for details).

$$\langle s_x \rangle = \left(\cos^2(\omega' t/2) - \frac{(\omega_0 - \omega)^2 - \Omega_l^2}{\omega'^2} \sin^2(\omega' t/2)\right) \cos\omega t - 2\left(\frac{\omega_0 - \omega}{\omega'} \cos(\omega' t/2) \sin(\omega' t/2)\right) \sin\omega t$$

$$\langle s_y \rangle = -\left(\cos^2(\omega' t/2) - \frac{(\omega_0 - \omega)^2 - \Omega_l^2}{\omega'^2} \sin^2(\omega' t/2)\right) \sin\omega t - 2\left(\frac{\omega_0 - \omega}{\omega'} \cos(\omega' t/2) \sin(\omega' t/2)\right) \cos\omega t$$

$$\langle s_z \rangle = \frac{1}{2}(|a|^2 - |b|^2) = \frac{(\omega - \omega_0)\Omega_l}{\omega'^2} \sin^2(\omega' t/2)$$

The projection of the polarization on the direction of motion

$$\hat{\boldsymbol{\beta}} \cdot s = \left(\cos^2(\omega' t/2) - \frac{1}{\omega'^2}[(\omega - \omega_0)^2 - \Omega_l^2] \sin^2(\omega' t/2)\right) \tag{56}$$

is shown in Figure 9.

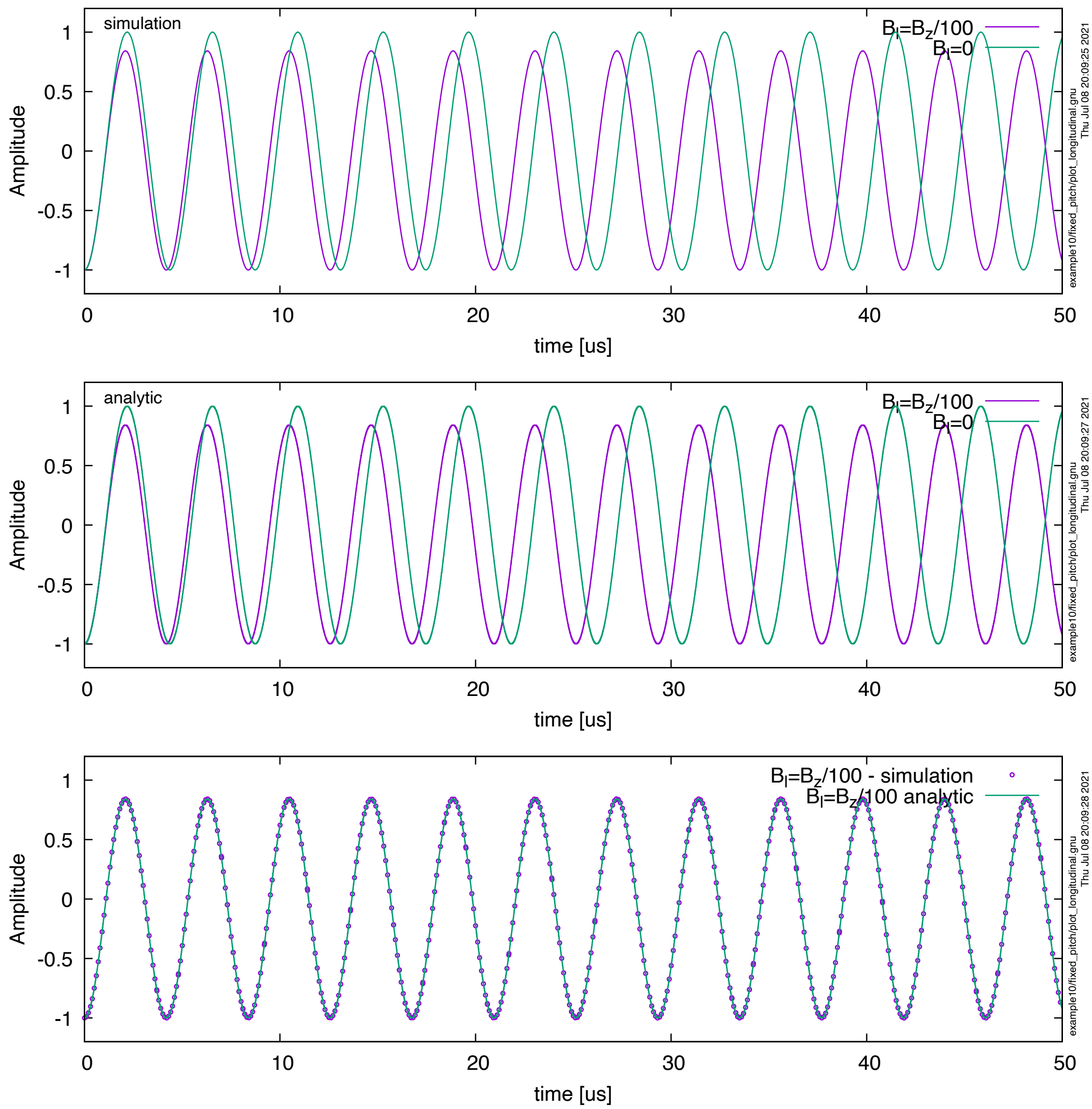


FIG. 9: The projection of the polarization on the direction of motion ($\hat{\boldsymbol{\beta}} \cdot \mathbf{s}$) with longitudinal magnetic field 10% of the vertical field and longitudinal field zero is shown in each of the three plots. The top plot is computed in simulation by integration of the equations of motion and the BMT equation. The middle plot is the analytic result (Equation 56). The bottom plot is the simulation and analytic superimposed. The agreement is excellent.

## B. Longitudinal Harmonics

In general the longitudinal field can be expanded in fourier harmonics. Then each harmonic is written as

$$\mathbf{B}_n = b_n \cos n\theta(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{y}}) + B_z\hat{\mathbf{z}}$$

where $n = 0$ is a uniform longitudinal field everywhere tangent to the trajectory and

$$\mathbf{B}_n \cdot \boldsymbol{\beta} = b_n \cos n\omega t$$

$$\mathcal{B} = (a_\mu + \frac{1}{\gamma})B_z\hat{\mathbf{z}} + b_n \cos n\omega t \left( (\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})(a_\mu + \frac{1}{\gamma}) - a_\mu \frac{\gamma}{\gamma+1}\beta^2 \cos(n\omega t)(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}}\right.$$

$$\boldsymbol{\sigma}\cdot\mathcal{B} = \begin{pmatrix} (a_\mu + \frac{1}{\gamma})B_z & b_n \left((a_\mu + \frac{1}{\gamma})\cos(n\omega t)e^{i\omega t} - a_\mu\frac{\gamma}{\gamma+1}\beta^2 \cos\right. \\ b_n\left((a_\mu + \frac{1}{\gamma})\cos(n\omega t)e^{-i\omega t} - a_\mu\frac{\gamma}{\gamma+1}\beta^2\cos(n\omega t)e^{-i\omega t}\right) & -(a_\mu + \frac{1}{\gamma})B_z \end{pmatrix}$$

$$\frac{e\hbar}{mc}\frac{\boldsymbol{\sigma}}{2}\cdot\mathcal{B} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{-i\omega t} \\ \Omega^* e^{i\omega t} & -\omega_0 \end{pmatrix}$$

where

$$\omega_0 = B_z \frac{e}{mc}(a_\mu + \frac{1}{\gamma})$$

$$\Omega = \frac{e}{mc}b_n\left((a_\mu + \frac{1}{\gamma})\cos(n\omega t) - a_\mu\frac{\gamma}{\gamma+1}\beta^2\cos(n\omega t)\right)$$

$$= \frac{1}{2}Xb_n\left(e^{-in\omega t} + e^{in\omega t}\right)$$

where

$$X = \frac{e}{mc}(a_\mu + \frac{1}{\gamma} - a_\mu\frac{\gamma}{\gamma+1}\beta^2).$$

Transforming to the rotating frame

$$\mathcal{H} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & \Omega \\ \Omega^* & -\omega_0 + \omega \end{pmatrix} \qquad (57)$$

In view of the time dependence of $\Omega$, we proceed using time dependent perturbation theory. Solutions to the unperturbed Hamiltonian $\mathcal{H}_0 = \frac{\hbar}{2}\begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}$, with $\eta = \omega_0 - \omega$ are

$$a(t) = a_0 e^{-i\eta t/2}$$

$$b(t) = b_0 e^{i\eta t/2}$$

The general solution to $\mathcal{H}$ is

$$
\begin{aligned}
\psi(t) &= a(t)\psi_a + b(t)\psi_b \\
&= c_a(t)e^{-i\eta t/2}\psi_a + c_b(t)e^{i\eta t/2}\psi_b
\end{aligned}
$$

Suppose in the initial state that $c_a(0) = c_b(0) = \frac{1}{\sqrt{2}}$. Then in zeroth order $c_a^{(0)}(t) = \frac{1}{\sqrt{2}}, c_b^{(0)}(t) = \frac{1}{\sqrt{2}}$. In first order

$$
\begin{aligned}
c_b^{(1)}(t) &= -i\int_0^t c_a^{(0)}\Omega^* e^{i\eta t'}dt' = -i\frac{1}{\sqrt{2}}\int_0^t \frac{1}{2}Xb_n\left(e^{-i(n\omega-\eta)t'} + e^{i(n\omega+\eta)t'}\right)dt' \\
&= -i\frac{1}{\sqrt{2}}\frac{1}{2}Xb_n\left(\frac{e^{-i(n\omega-\eta)t}-1}{-i(n\omega-\eta)} + \frac{e^{i(n\omega+\eta)t}-1}{i(n\omega+\eta)}\right) \\
c_a^{(1)}(t) &= -i\int_0^t c_b^{(0)}\Omega e^{-i\eta t'}dt'
\end{aligned}
$$

In second order

$$
\begin{aligned}
c_b^{(2)}(t) &= -i\int_0^t c_a^{(1)}(t')\Omega^* e^{i\eta t'}dt' \\
c_a^{(2)}(t) &= -i\int_0^t c_b^{(1)}(t')\Omega e^{-i\eta t'}dt' \\
&= -\frac{1}{4\sqrt{2}}\int_0^t X^2 b_n^2\left(\frac{e^{-i(2n\omega-\eta)t'} - e^{-in\omega t'}}{-i(n\omega-\eta)} + \frac{e^{i(2n\omega+\eta)t'} - e^{in\omega t'}}{i(n\omega+\eta)}\right. \\
&\quad \left. + \frac{e^{i\eta t'} - e^{in\omega t'}}{-i(n\omega-\eta)} + \frac{e^{i\eta t'} - e^{-in\omega t'}}{i(n\omega-\eta)}\right)e^{-i\eta t'}dt' \\
&= -\frac{1}{4\sqrt{2}}\int_0^t X^2 b_n^2\left(\frac{e^{-i(2n\omega)t'} - e^{-i(n\omega+\eta)t'}}{-i(n\omega-\eta)} + \frac{e^{i(2n\omega)t'} - e^{i(n\omega-\eta)t'}}{i(n\omega+\eta)}\right. \\
&\quad \left. + \frac{1 - e^{i(n\omega-\eta)t'}}{-i(n\omega-\eta)} + \frac{1 - e^{-i(n\omega+\eta)t'}}{i(n\omega+\eta)}\right)dt'
\end{aligned}
$$

$$
\begin{aligned}
c_a^{(2)}(t) &= -\frac{1}{4\sqrt{2}}\int_0^t X^2 b_n^2 \left(\frac{e^{-i(2n\omega)t'} - e^{-i(n\omega+\eta)t'}}{-i(n\omega-\eta)} + \frac{e^{i(2n\omega)t'} - e^{i(n\omega-\eta)t'}}{i(n\omega+\eta)} + \frac{1-e^{i(n\omega-\eta)t'}}{-i(n\omega-\eta)} + \frac{1-e^{-i(n\omega+\eta)t'}}{i(n\omega+\eta)}\right) dt' \\
&= -\frac{1}{4\sqrt{2}} X^2 b_n^2 \left(\frac{1}{-i(n\omega-\eta)}\left(\frac{e^{-i(2n\omega)t}-1}{-2in\omega} - \frac{e^{-i(n\omega+\eta)t}-1}{-i(n\omega+\eta)} + t - \frac{e^{i(n\omega-\eta)t}-1}{i(n\omega-\eta)}\right)\right. \\
&\quad \left. + \frac{1}{i(n\omega+\eta)}\left(\frac{e^{i(2n\omega)t}-1}{2in\omega} - \frac{e^{i(n\omega-\eta)t}-1}{i(n\omega-\eta)} + t - \frac{e^{-i(n\omega+\eta)t}-1}{-i(n\omega+\eta)}\right)\right) \qquad (58) \\
&= -\frac{1}{4\sqrt{2}} X^2 b_n^2 \left(\frac{1}{-i(n\omega-\eta)}\left(\frac{e^{-in\omega t}\sin(n\omega)t}{n\omega} - \frac{e^{-i(n\omega+\eta)t}-1}{-i(n\omega+\eta)} + t - \frac{e^{i(n\omega-\eta)t}-1}{i(n\omega-\eta)}\right)\right. \\
&\quad \left. + \frac{1}{i(n\omega+\eta)}\left(\frac{e^{in\omega t}\sin(n\omega)t}{n\omega} - \frac{e^{i(n\omega-\eta)t}-1}{i(n\omega-\eta)} + t - \frac{e^{-i(n\omega+\eta)t}-1}{-i(n\omega+\eta)}\right)\right) \qquad (59) \\
&= \frac{1}{\sqrt{2}}\left(-i\frac{\lambda}{2}t + C_{a-rem}^{(2)}\right)
\end{aligned}
$$

where $\lambda$ is the coefficient of imaginary time $it$ in Equation 59

If $n = 0$,

$$\lambda(n=0) = 2X^2 b_0^2 \frac{1}{\eta},$$

as the terms proportional to $\frac{\sin(n\omega t)}{n\omega}$ reduce to $t$ when $n = 0$.

If $n > 0$,

$$\lambda(n>0) = X^2 b_n^2 \frac{-\eta}{(n\omega)^2 - \eta^2}$$

Next recall

$$
\begin{aligned}
c_a(t) &\sim \left(c_a^{(0)} + c_a^{(1)} + c_a^{(2)} + \ldots\right) \\
&= \frac{1}{\sqrt{2}}\left(1 + c_a^{(1)} - i\frac{\lambda}{2}t + C_{a-rem}^{(2)}\right) \\
&\sim \frac{1}{\sqrt{2}}\left(1 + c_a^{(1)} + C_{a-rem}^{(2)}\right) e^{-i\lambda t/2}
\end{aligned}
$$

Note that there is no contribution to a frequency shift from $c_a^{(0)}$ or $c_a^{(1)}$ since they contain no term proportional to $t$, whereas $c_a^{(2)}$ does have such a term, with coefficient $\lambda/2$. $C_{a-rem}^{(2)}$ is all of the terms remaining in 59 that are not coefficients of $it$, which in turn is different for $n = 0$ and $n > 0$. Fortunately the value of that remaining term does not contribute to the frequency shift.

The amplitudes to be in the states $\psi_a$ and $\psi_b$ are

$$
\begin{aligned}
a(t) &= c_a(t)e^{-i\eta t/2} = \frac{1}{\sqrt{2}}\left(1 + c_a^{(1)} + C_{a-rem}^{(2)}\right) e^{-i(\eta+\lambda)t/2} \\
b(t) &= c_b(t)e^{i\eta t/2} = \frac{1}{\sqrt{2}}\left(1 + c_b^{(1)} + C_{b-rem}^{(2)}\right) e^{i(\eta+\lambda)t/2}
\end{aligned}
$$

As shown in 6, the polarization $\langle\psi \mid \boldsymbol{\sigma}/2 \mid \psi\rangle$ in the x and y directions is proportional to the real and imaginary parts of $a^*b$.

$$a^*b = \frac{1}{2}\left(1 + c_a^{(1)} + C_{a-rem}^{(2)}\right)^*\left(1 + c_b^{(1)} + C_{b-rem}^{(2)}\right)e^{i(\eta+\lambda)t}$$

For $n = 0$, the precession frequency in the rotating frame at the magic momentum is

$$\begin{aligned}
\omega' &= \eta + \lambda(n) = \omega_0 - \omega + \lambda(n) \\
&= \omega_0 - \omega + 2\frac{X^2 b_n^2(\omega_0 - \omega)}{((n\omega)^2 - (\omega_0 - \omega)^2)} \\
&= \omega_0 - \omega + \frac{1}{2}\left(\frac{e}{mc}a_\mu\right)^2\left(1 + \frac{\gamma^2 - 1}{\gamma} - \frac{\gamma - 1}{\gamma}\right)^2\frac{1}{(\omega_0 - \omega)} \\
&= \omega_a + \frac{1}{2}\omega_a^2\frac{b_0^2}{B_z^2}\gamma^2\frac{1}{\omega_a} \\
&= \omega_a(1 + \frac{1}{2}\frac{b_0^2}{B_z^2}\gamma^2) \\
\frac{\Delta\omega'}{\omega_a} &= \frac{1}{2}\frac{b_0^2}{B_z^2}\gamma^2
\end{aligned}$$

consistent (to second order) with the exact result in Equation 55. (Note that for clarity we here use the definition $\omega_a \equiv \frac{eB_z}{mc}a_\mu$ rather than $\omega_a \equiv \frac{e|\mathbf{B}|}{mc}a_\mu$.) If $n > 0$ the precession frequency in the rotating frame at the magic momentum is

$$\begin{aligned}
\omega' &= \omega_a - \omega_a^2\frac{b_n^2}{B_z^2}\frac{\gamma^2}{4}\frac{\omega_a}{((n\omega)^2 - \omega_a^2)} \\
&= \omega_a\left(1 - \frac{b_n^2}{B_z^2}\frac{\gamma^2}{4}\frac{\omega_a^2}{(n\omega)^2 - \omega_a^2}\right) \\
\frac{\Delta\omega'}{\omega_a} &= -\frac{1}{4}\frac{b_n^2}{B_z^2}\gamma^2\frac{\omega_a}{(n\omega)^2 - \omega_a^2}
\end{aligned}$$

It is amusing to see how the contribution to the precession frequency depends on the harmonic number. To that end consider the ratio of the $n > 1$ harmonics to the $n = 0$ harmonic

$$\frac{\Delta\omega'(n > 0)}{\Delta\omega'(n = 0)} = \frac{1}{2}\frac{\omega_a^2}{(n\omega)^2 - \omega_a^2}\frac{b_n^2}{b_0^2}.$$

And if $b_n = b_0$

$$\frac{\Delta\omega'(n > 0)}{\Delta\omega'(n = 0)} = \frac{1}{2}\frac{\omega_a^2}{(n\omega)^2 - \omega_a^2}$$

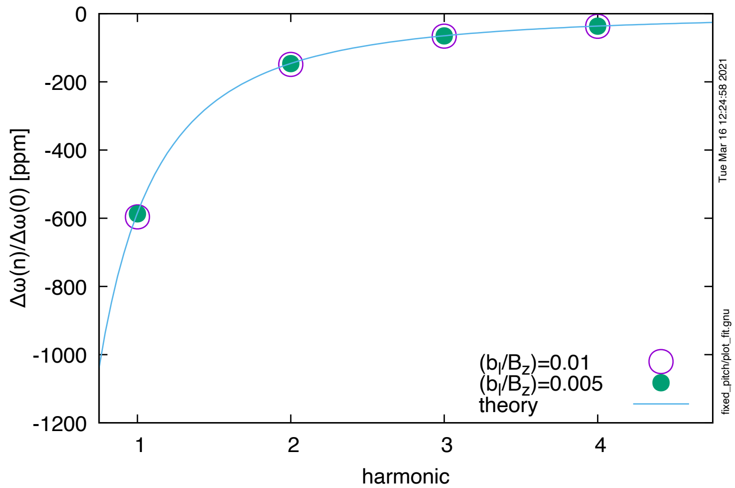


FIG. 10: $\Delta\omega(n)$ is the frequency shift for the $n^{th}$ harmonic. $\Delta\omega(0)$ is the frequency shift for the $n = 0$ harmonic. The ratio is shown as a function of $n$. $b_n = b_0$ for all $n$. The points are from integration of the Thomas-BMT equation and the equations of motion. The open circles are computed for $b_l/B_z = 0.01$ and the filled circles for $b_l/B_z = 0.005$. The line is Equation 60 where $n$ is treated as a continuous variable.

Using $\eta = \frac{e}{mc}B_z a_\mu$ and $\omega = \frac{e}{mc\gamma}B_z$ so that we write $\omega = \frac{\eta}{\gamma a_\mu}$

$$\begin{aligned}
\Delta\omega'(n>0) &= \frac{1}{2}\frac{-(\gamma a_\mu \omega)^2}{(n\omega)^2 - \omega^2(\gamma a_\mu)^2}\Delta\omega'(n=0) \\
&= \frac{1}{2}\frac{-(\gamma a_\mu \omega)^2}{(n\omega)^2 - \omega^2(\gamma a_\mu)^2}\Delta\omega'(n=0) \\
&= \frac{1}{2}\frac{-(\gamma a_\mu)^2}{n^2 - (\gamma a_\mu)^2}\Delta\omega'(n=0) \\
&= \frac{1}{2}\frac{-\gamma^2}{n^2(\gamma^2-1)^2 - \gamma^2}\Delta\omega'(n=0) \qquad (60)
\end{aligned}$$

(Note that the tuneshift for $n > 0$ harmonics has the opposite sign of the $n = 0$ term.) The dependence of frequency shift on longitudinal harmonic is shown in Figure 10. The points in the plot are from simulation (integration of the Thomas-BMT equation).

## IX. ELECTRIC FIELD CORRECTION

The magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$ and the electric field is in the radial direction We assume no betatron oscillations so the particle circulates on its closed orbit. The electric field is

$$\mathbf{E} = E_r(\sin\theta\ \hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}). \qquad (61)$$

The velocity has the form

$$\boldsymbol{\beta} = \beta(\cos\omega t\ \hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}}) \tag{62}$$

Substitution into the Lorentz force law (Equation 16 gives us

$$\begin{aligned}
-\omega\beta(\sin\omega t\ \hat{\mathbf{x}} + \cos\omega t\hat{\mathbf{y}}) &= \frac{e}{mc\gamma}\left(E_r(\sin\omega t\ \hat{\mathbf{x}} + \cos\omega t\hat{\mathbf{y}}) - \beta B_z(\sin\omega t\ \hat{\mathbf{x}} + \cos\omega t\hat{\mathbf{y}})\right) \\
\rightarrow \omega &= \frac{e}{mc\gamma}\left(\frac{E_r}{\beta} - B_z\right)
\end{aligned} \tag{63}$$

The effective magnetic field

$$\boldsymbol{\mathcal{B}} = \left[B_z(a_\mu + \frac{1}{\gamma}) - \left(a_\mu - \frac{1}{\gamma+1}\right)\beta E_r\right]\hat{\mathbf{z}} \tag{64}$$

and the Hamiltonian

$$\mathcal{H} = -\frac{\hbar}{2}\begin{pmatrix}\omega_0 & 0 \\ 0 & -\omega_0\end{pmatrix}$$

where

$$\omega_0 = \frac{e}{mc}\left(B_z(a_\mu + \frac{1}{\gamma}) - \left(a_\mu - \frac{1}{\gamma+1}\right)\beta E_r\right)$$

In the rotating frame

$$\mathcal{H}_{rot} = -\frac{\hbar}{2}\begin{pmatrix}\omega_0 - \omega & 0 \\ 0 & \omega - \omega_0\end{pmatrix} \tag{65}$$

with $\omega$ given in Equation 63. The precession frequency in the rotating frame

$$\begin{aligned}
\omega' &= \omega_0 - \omega \\
&= \frac{e}{mc}\left(B_z a_\mu - \left(a_\mu - \frac{1}{\gamma+1} + \frac{1}{\beta^2\gamma}\right)\beta E_r\right) \\
&= \frac{e}{mc}\left(B_z a_\mu - \left(a_\mu - \frac{1}{\gamma+1} + \frac{\gamma}{\gamma^2-1}\right)\beta E_r\right) \\
&= \frac{e}{mc}\left(B_z a_\mu - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\beta E_r\right)
\end{aligned}$$

The contribution to the precession frequency in the rotating frame vanishes at the magic momentum.

## X. ZERO MAGNETIC FIELD

It is instructive to study the case of an electrostatic focusing channel in the absence of a magnetic field. We expect that at the magic momentum, the projection of polarization on velocity $\boldsymbol{\beta} \cdot \mathbf{s}$ will be constant. Let's see how that happens.

Now $\omega_0 = \omega = 0$, and $\beta_{xy} \to \beta_x$).

$$
\begin{aligned}
\mathcal{H} &= \begin{pmatrix} 0 & i\Pi(t) \\ -i\Pi(t) & 0 \end{pmatrix} \\
&= R \begin{pmatrix} \Pi(t) & 0 \\ 0 & -\Pi(t) \end{pmatrix} R^{-1}
\end{aligned}
$$

where

$$
R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{i}{2} R \begin{pmatrix} \Pi(t) & 0 \\ 0 & -\Pi(t) \end{pmatrix} R^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \\
\frac{\partial}{\partial t} R^{-1} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{i}{2} R \begin{pmatrix} \Pi(t) & 0 \\ 0 & -\Pi(t) \end{pmatrix} R^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \\
\frac{\partial}{\partial t} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} &= \frac{i}{2} \begin{pmatrix} \Pi(t) & 0 \\ 0 & -\Pi(t) \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}
\end{aligned}
$$

The now decoupled Schrodinger equations are straightforward to solve.

$$
\begin{aligned}
\int \frac{\dot{\bar{a}}}{\bar{a}} dt &= \frac{i}{2} \Pi(t)\ dt = \frac{i}{2} \frac{e}{mc} \left( a_\mu + \frac{1}{\gamma + 1} \right) E_z \beta_x(t)\ dt \\
\ln \bar{a} &= i\kappa \frac{ekz_0}{2mc} \int \cos \omega_v t \beta_x^0 \cos(\beta_z^0 \sin \omega_v t)\ dt \\
&\sim i\kappa \frac{ekz_0}{2mc} \int \cos \omega_v t \beta_x^0 (1 - \frac{1}{2} (\beta_z^0 \sin \omega_v t)^2)\ dt \\
&\sim i\kappa \frac{ekz_0}{2mc\omega_v} \beta_x^0 \left( \sin \omega_v t - \frac{1}{3!} \beta_z^{02} \sin^3 \omega_v t) \right) \\
&\sim \frac{i}{2} \kappa \gamma_0 \beta_z^0 \beta_x^0 (\sin \omega_v t)
\end{aligned}
$$

where $\kappa = \left(a_\mu + \frac{1}{\gamma+1}\right)$ and we keep to order $\beta_z^2$ or equivalently $z_0^2$. Then

$$\bar{a} \sim \bar{a}_0 \exp(\frac{i}{2}\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t)$$
$$\bar{b} \sim \bar{b}_0 \exp(-\frac{i}{2}\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t)$$

Recall

$$\begin{pmatrix} a \\ b \end{pmatrix} = R \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix}$$

and

$$\begin{aligned}
\langle s_x \rangle &= \begin{pmatrix} a^* & b^* \end{pmatrix} \sigma_x \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} \bar{a}^* & \bar{b}^* \end{pmatrix} R^\dagger \sigma_x R \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \bar{a}^* & \bar{b}^* \end{pmatrix} \sigma_x \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} \\
&= \bar{a}^*\bar{b} + \bar{b}^*\bar{a} \\
\langle s_y \rangle &= \begin{pmatrix} \bar{a}^* & \bar{b}^* \end{pmatrix} R^\dagger \sigma_y R \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = -|\bar{a}|^2 + |\bar{b}|^2 \\
\langle s_z \rangle &= \begin{pmatrix} \bar{a}^* & \bar{b}^* \end{pmatrix} R^\dagger \sigma_z R \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = |\bar{a}|^2 - |\bar{b}|^2 - i(\bar{a}^* b - \bar{b}^* a)
\end{aligned}$$

If we assume $\bar{a}_0 = \bar{b}_0$, so that the polarization is initially in the $x$-direction,

$$\begin{aligned}
\langle s_x \rangle &= \cos(\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t) \\
\langle s_y \rangle &= 0 \\
\langle s_z \rangle &= \sin(\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t)
\end{aligned} \tag{66}$$

Using 22, 23, and 66 we find that

$$\mathbf{s}\cdot\hat{\boldsymbol{\beta}} = \frac{\beta_x^0 \cos(\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t)\cos(\beta_z^0 \sin\omega_v t) + \sin(\kappa\gamma_0\beta_z^0\beta_x^0 \sin\omega_v t)\sin(\beta_z^0 \sin\omega_v t)}{[{\beta_x^0}^2 \cos^2(\beta_z^0 \sin\omega_v t) + \sin^2(\beta_z^0 \sin\omega_v t)]^{1/2}}$$

The rate of change of $\mathbf{s}\cdot\hat{\boldsymbol{\beta}}$ will include even harmonics of $\omega_v$. At the magic momentum, $\mathbf{s}\cdot\hat{\boldsymbol{\beta}}$ is independent of time ($\kappa\gamma_0\beta_x^0 \sim 1$). (See XII E for details.)

## XI. SUMMARY

**Fixed pitch**, with velocity $\beta_z$ parallel to the magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$.

$$\omega' = \omega_a(1 - \beta_z^2)^{1/2} \tag{67}$$

**Time dependent pitch.** Vertical betatron motion with amplitude $\beta_z^0$. ($\beta_z^0 \sim \psi$ where $\psi$ is the pitch angle) for arbitrary (but non zero) vertical betatron frequency $\omega_v$ and at the magic momentum. To order $(\beta_z^0)^2$

$$\omega' = \omega_a \left(1 - \frac{(\beta_z^0)^2}{2}\left[\frac{\gamma}{\gamma+1} - \frac{1}{2}\omega_a\beta_{xy}^2\left(\frac{2\omega_v(\frac{\gamma^2}{\gamma+1}\frac{\omega_v}{\omega}) - \omega_a(\frac{\gamma^2}{(\gamma+1)^2} + \gamma^2\frac{\omega_v^2}{\omega^2})}{\omega_v^2 - \omega_a^2}\right)\right]\right)$$

**Time dependent pitch where pitching frequency is much greater than** $\omega_a$. Vertical betatron motion with amplitude $\beta_z^0$ in the $\omega_v \gg \omega_a$ limit, at the magic momentum

$$\omega' = \omega_a\left(1 - \frac{(\beta_z^0)^2}{4}\left[\frac{\gamma^2}{(\gamma^2-1)}\right]\right)$$

**Radial magnetic field**

$$\omega' = \omega_a\left[1 + \left(\frac{B_r}{B_z}\right)^2\left(\frac{1 + 1/a_\mu}{\gamma^2}\right)^2\right]^{1/2} \tag{68}$$

**Uniform longitudinal magnetic field**

$$\omega' = \omega_a\left[1 + \left(\frac{B_l}{B_z}\right)^2\left(\frac{1}{a_\mu\gamma} + \frac{1}{\gamma}\right)^2\right]^{1/2} \tag{69}$$

**Nonuniform longitudinal field in terms of fourier harmonics** to order $(b_n/B_z)^2$ for magnetic field harmonics and $b_n$ defined by

$$\mathbf{B}_n = b_n \cos n\theta(\cos\theta\ \hat{\mathbf{x}} - \sin\theta\ \hat{\mathbf{y}}) + B_z\hat{\mathbf{z}}$$

$$\omega' = \omega_a\left(1 - \frac{b_n^2}{B_z^2}\frac{\gamma^2}{4}\frac{\omega_a^2}{(n\omega)^2 - \omega_a^2}\right)$$
$$\frac{\Delta\omega'}{\omega_a} = -\frac{1}{4}\frac{b_n^2}{B_z^2}\gamma^2\frac{\omega_a}{(n\omega)^2 - \omega_a^2}$$

**Closed orbit electric field contribution**

$$\omega' = \frac{e}{mc}\left(B_z a_\mu - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\beta E_r\right) \tag{70}$$

## XII. APPENDICES

### A. Solution of Schrodinger Equation in lab frame

We aim to solve the coupled differential equations exactly.

$$\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} a(t)\omega_0 + b(t)\Omega e^{i\omega t} \\ a(t)\Omega e^{-i\omega t} - b(t)\omega_0 \end{pmatrix}.$$

To solve these coupled differential equations first differentiate the $\dot{b}(t)$ equation,

$$\ddot{b} = \frac{i}{2}\left[(\dot{a} - i\omega a)\Omega e^{-i\omega t} - \dot{b}\omega_0\right]$$

then substitute for $\dot{a}(t)$. We have

$$\ddot{b} = \frac{i}{2}\left[(\frac{i}{2}(a\omega_0 + b\Omega e^{i\omega t}) - i\omega a)\Omega e^{-i\omega t} - \dot{b}\omega_0\right]$$

and

$$\ddot{b} = \frac{i}{2}\left[(ia\Omega e^{-i\omega t}(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

Then substitute the expression for $a(t)$ from the original equation

$$a(t)\Omega e^{-i\omega t} = -2i\dot{b}(t) + b(t)\omega_0.$$

and we get

$$\ddot{b} = \frac{i}{2}\left[(i\left((-2i\dot{b} + b\omega_0)e^{i\omega t}/\Omega\right)\Omega e^{-i\omega t}(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

$$\ddot{b} = \frac{i}{2}\left[(i\left(-2i\dot{b} + b\omega_0\ \right)(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

$$\ddot{b} = \frac{i}{2}\left[(-2\dot{b}\omega + i\frac{b}{2}\left(\omega_0^2 - 2\omega_0\omega + \Omega^2\right)\right]$$

$$\ddot{b} = -i\dot{b}\omega - \frac{b}{4}\left(\omega_0^2 - 2\omega_0\omega + \Omega^2\right)$$

$$\ddot{b} = -i\dot{b}\omega - \frac{b}{4}\left(\omega'^2 - \omega^2\right)$$

where

$$\omega_0^2 - 2\omega_0\omega + \Omega^2 = \omega'^2 - \omega^2$$
$$\rightarrow \omega'^2 = (\omega_0 - \omega)^2 + \Omega^2 \tag{71}$$

Define $\gamma = i\omega$, and $\alpha = \frac{b}{4}(\omega'^2 - \omega^2)$. Now our coupled second order differential equation looks like the equation of motion for a damped harmonic oscillator.

$$\ddot{b} = -\gamma \dot{b} - \alpha b$$

The general solution has the form

$$b = Ae^{i\theta t}$$

Substitution into the differential equation gives

$$-\theta^2 = -i\theta\gamma - \alpha$$

and

$$\theta = \frac{1}{2}\left(i\gamma \pm \sqrt{-\gamma^2 + 4\alpha}\right) = \frac{1}{2}\left(-\omega \pm \sqrt{\omega^2 + (\omega'^2 - \omega^2)}\right) = -\frac{\omega}{2} \pm \frac{\omega'^2}{2}$$

so

$$b = \left[Ce^{i\frac{\omega'}{2}t} + De^{-i\frac{\omega'}{2}t}\right] e^{-i\frac{\omega}{2}t}$$

or even better

$$b = \left[A\cos(\omega' t/2) + B\sin(\omega' t/2)\right] e^{-i\frac{\omega}{2}t}$$

The boundary conditions give us

$$b(0) = A = b_0$$

and

$$\begin{aligned}
\dot{b}(0) &= (-i\omega/2)A + (\omega'/2)B = \frac{i}{2}(\Omega a_0 - \omega_0 b_0) \\
&\to \omega b_0 + i\omega' B = (\omega_0 b_0 - \Omega a_0) \\
&\to B = \frac{i}{\omega'}\left[(\omega - \omega_0)b_0 + \Omega a_0\right]
\end{aligned}$$

Then

$$b = \left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega - \omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right] e^{-i\frac{\omega}{2}t}$$

and

$$
\begin{aligned}
a &= (-2i\dot{b} + b\omega_0)\frac{e^{i\omega t}}{\Omega} \\
&= \Bigg[\left(-2i(-\frac{b_0\omega'}{2}\sin(\omega' t/2) + \frac{i}{2}[(\omega-\omega_0)b_0 + \Omega a_0]\cos(\omega' t/2)\right) e^{-i\omega t/2}) \\
&+ -2\frac{\omega}{2}\left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega-\omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right] e^{-i\frac{\omega}{2}t} \\
&+ \omega_0\left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega-\omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right] e^{-i\frac{\omega}{2}t}\Bigg]\frac{e^{i\omega t}}{\Omega} \\
&= \Bigg(-2i(-\frac{b_0\omega'}{2}\sin(\omega' t/2) + \frac{i}{2}[(\omega-\omega_0)b_0 + \Omega a_0]\cos(\omega' t/2)) \\
&+ (\omega_0-\omega)\left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega-\omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right]\Bigg)\frac{e^{i\frac{\omega}{2}t}}{\Omega} \\
&= ((ib_0\omega'\sin(\omega' t/2) + [(\omega-\omega_0)b_0 + \Omega a_0]\cos(\omega' t/2) \\
&+ (\omega_0-\omega)\left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega-\omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right]\Bigg)\frac{e^{i\frac{\omega}{2}t}}{\Omega} \\
&= ((ib_0\omega'\sin(\omega' t/2) + [\Omega a_0]\cos(\omega' t/2) \\
&+ (\omega_0-\omega)\left[\frac{i}{\omega'}\left[(\omega-\omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right]\Bigg)\frac{e^{i\frac{\omega}{2}t}}{\Omega} \\
&= \left[(ib_0\omega' - ib_0\frac{(\omega-\omega_0)^2}{\omega'})\sin(\omega' t/2)) + a_0\Omega(\cos(\omega' t/2) + \frac{i(\omega_0-\omega)}{\omega'}\sin(\omega' t/2))\right]\frac{e^{i\omega t/2}}{\Omega} \\
&= \left[(-ib_0\frac{\Omega^2}{\omega'})\sin(\omega' t/2)) + a_0\Omega(\cos(\omega' t/2) + \frac{i(\omega_0-\omega)}{\omega'}\sin(\omega' t/2))\right]\frac{e^{i\omega t/2}}{\Omega} \\
&= \left[(-ib_0\frac{\Omega}{\omega'})\sin(\omega' t/2)) + a_0(\cos(\omega' t/2) + \frac{i(\omega_0-\omega)}{\omega'}\sin(\omega' t/2))\right] e^{i\omega t/2} \\
&= \left[a_0\cos(\omega' t/2) + \frac{i}{\omega'}[(\omega_0-\omega)a_0 - \Omega b_0]\sin(\omega' t/2)\right] e^{i\omega t/2}
\end{aligned}
$$

or

$$
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \left(\cos(\omega' t/2) - \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) e^{i\frac{\omega}{2}t} & -\frac{i\Omega}{\omega'}\sin(\omega' t/2)e^{i\frac{\omega}{2}t} \\ -\frac{i\Omega}{\omega'}\sin(\omega' t/2)e^{-i\frac{\omega}{2}t} & \left(\cos(\omega' t/2) + \frac{i(\omega-\omega_0)}{\omega'}\sin(\omega' t/2)\right) e^{-i\frac{\omega}{2}t} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{72}
$$

### B. Linear Albegra

Determine the eigenvalues and eigenvectors of a Hermitian matrix

$$
H = \begin{pmatrix} A & B \\ B^* & -A \end{pmatrix}
$$

The eigenvalues

$$0 = (A - \lambda)(-A - \lambda) - |B|^2$$
$$\rightarrow \lambda_{\pm} = \pm\sqrt{A^2 + |B|^2}$$

but for conevience define $\lambda = +\sqrt{A^2 + |B|^2}$. The eigenvector equation

$$\begin{pmatrix} A & B \\ B^* & -A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$
$$Aa + Bb = \lambda a, \quad B^* a - Ab = \lambda b$$
$$b(Aa + Bb) = a(B^* a - Ab)$$
$$0 = Bb^2 + 2abA - a^2 B^*$$
$$\rightarrow a = \frac{2Ab \pm \sqrt{(2bA)^2 + 4|B|^2 b^2}}{-2B^*}$$
$$a = \frac{b}{B^*}\left(A \pm \sqrt{(A)^2 + |B|^2}\right) = \frac{b}{B^*}(A \pm \lambda)$$

Normalize

$$|b|^2 + |a|^2 = 1 = |b|^2 \left(\frac{(A+\lambda)^2}{|B|^2} + 1\right)$$
$$\rightarrow |b| = b = e^{i\theta} \left(\frac{|B|^2}{(A+\lambda)^2 + |B^2|}\right)^{1/2}$$
$$a = \frac{e^{i\theta}|B|}{B^*} \frac{(A+\lambda)}{\sqrt{(A+\lambda)^2 + |B^2|}}$$
$$a = \frac{(A+\lambda)}{\sqrt{(A+\lambda)^2 + |B^2|}}$$

That last step follows if we choose $a$ to be real which requires that $B^* = |B| e^{i\theta}$.

Now let's check to be sure

$$\frac{1}{N}\begin{pmatrix} A & B \\ B^* & -A \end{pmatrix} \begin{pmatrix} (A+\lambda) \\ e^{i\theta}|B| \end{pmatrix} = \lambda \frac{1}{N} \begin{pmatrix} (A+\lambda) \\ e^{i\theta}|B| \end{pmatrix}$$
$$\begin{pmatrix} A(A+\lambda) + B|B|e^{i\theta} \\ (A+\lambda)B^* - A|B|e^{i\theta} \end{pmatrix} = \lambda \begin{pmatrix} (A+\lambda) \\ e^{i\theta}|B| \end{pmatrix}$$

where $N = ((A+\lambda)^2 + |B^2|)^{1/2} = (2\lambda(\lambda + A))^{1/2}$. The orthogonal eigenvectors with eigenvalues $\pm\lambda$ are

$$v_+ = \frac{1}{N}\begin{pmatrix} A+\lambda \\ B^* \end{pmatrix}, \quad v_- = \frac{1}{N}\begin{pmatrix} B \\ -(A+\lambda) \end{pmatrix} \tag{73}$$

The matrix $H$ is diagonalized by the similarity transformation

$$S^{-1}HS = \tilde{H} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

where the matrix $S$ is constructed from the eigenvectors

$$S = \frac{1}{N}\begin{pmatrix} A+\lambda & B \\ B^* & -(A+\lambda) \end{pmatrix}$$

The general solution to the Schrodinger equation

$$\tilde{H}\psi = i\frac{\partial}{\partial t}\psi$$

are

$$\tilde{\psi} = \tilde{a}_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda t} + \tilde{b}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda t}$$

Transform to the lab basis

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = S \begin{pmatrix} \tilde{a}_0 e^{-i\lambda t} \\ \tilde{b}_0 e^{i\lambda t} \end{pmatrix} \equiv \mathcal{S}(t) \begin{pmatrix} \tilde{a}_0 \\ \tilde{b}_0 \end{pmatrix}$$

where

$$\mathcal{S}(t) = \begin{pmatrix} S_{11}e^{-i\lambda t} & S_{12}e^{i\lambda t} \\ S_{21}e^{-i\lambda t} & S_{22}e^{i\lambda t} \end{pmatrix}$$

Then

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \mathcal{S}(t) \begin{pmatrix} \tilde{a}_0 \\ \tilde{b}_0 \end{pmatrix} = \mathcal{S}(t)S^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

Since

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = S \begin{pmatrix} \tilde{a}_0 \\ \tilde{b}_0 \end{pmatrix}$$

in the lab basis

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \mathcal{S}(t) \begin{pmatrix} \tilde{a}_0 \\ \tilde{b}_0 \end{pmatrix} = \mathcal{S}(t)S^{-1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \frac{1}{N}\begin{pmatrix} (A+\lambda)e^{-i\lambda t} & Be^{i\lambda t} \\ B^*e^{-i\lambda t} & -(A+\lambda)e^{i\lambda t} \end{pmatrix} \frac{1}{N}\begin{pmatrix} A+\lambda & B \\ B^* & -(A+\lambda) \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{N^2} \begin{pmatrix} (A+\lambda)^2 e^{-i\lambda t} + |B|^2 e^{i\lambda t} & (A+\lambda)Be^{-i\lambda t} - B(A+\lambda)e^{i\lambda t} \\ (A+\lambda)B^* e^{-i\lambda t} - B^*(A+\lambda)e^{i\lambda t} & |B|^2 e^{-i\lambda t} + (A+\lambda)^2 e^{i\lambda t} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$$= \frac{1}{N^2} \begin{pmatrix} ((A+\lambda)^2 + |B|^2)\cos(\lambda t) + i(|B|^2 - (A+\lambda)^2)\sin(\lambda t) & -2i(A+\lambda)B\sin(\lambda t) \\ -2i(A+\lambda)B^*\sin(\lambda t) & ((A+\lambda)^2 + |B|^2)\cos(\lambda t) - i(|B|^2 - ( \end{pmatrix}$$

$$= \frac{1}{N^2} \begin{pmatrix} ((A+\lambda)^2 + |B|^2)\cos(\lambda t) + i(-2A(A+\lambda))\sin(\lambda t) & -2i(A+\lambda)B\sin(\lambda t) \\ -2i(A+\lambda)B^*\sin(\lambda t) & ((A+\lambda)^2 + |B|^2)\cos(\lambda t) + i(2A(A+\lambda)) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\lambda t) + i\frac{(-2A(A+\lambda))}{N^2}\sin(\lambda t) & -2i\frac{(A+\lambda)B}{N^2}\sin(\lambda t) \\ -2i\frac{(A+\lambda)B^*\sin(\lambda t)}{N^2} & \cos(\lambda t) + 2iA\frac{(A+\lambda))}{N^2}\sin(\lambda t) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

Then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\lambda t) + i\frac{-2A(A+\lambda)}{2\lambda(A+\lambda)}\sin(\lambda t) & -2i\frac{(A+\lambda)B}{2\lambda(A+\lambda)}\sin(\lambda t) \\ -2i\frac{(A+\lambda)B^*\sin(\lambda t)}{2\lambda(A+\lambda)} & \cos(\lambda t) + 2iA\frac{(A+\lambda))}{2\lambda(A+\lambda)}\sin(\lambda t) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\lambda t) - i\frac{A}{\lambda}\sin(\lambda t) & -i\frac{B}{\lambda}\sin(\lambda t) \\ -i\frac{B^*\sin(\lambda t)}{\lambda} & \cos(\lambda t) + i\frac{A}{\lambda}\sin(\lambda t) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

If $A = -(\omega_0 - \omega)/2$, $B = -\tilde{\Omega}/2$ and $\omega' = 2\lambda = \sqrt{(\omega_0 - \omega)^2 + |\tilde{\Omega}|^2}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\omega' t/2) - i\frac{\omega - \omega_0}{\omega'}\sin(\omega' t/2) & i\frac{\tilde{\Omega}}{\omega'}\sin(\omega' t/2) \\ i\frac{\tilde{\Omega}^*\sin(\omega' t/2)}{\omega'} & \cos(\omega' t/2) + i\frac{(\omega - \omega_0)}{\omega'}\sin(\omega' t/2) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

The above is the solution in the rotating frame. In the lab frame

$$\begin{pmatrix} a \\ b \end{pmatrix}_{lab} = R^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} (\cos(\omega' t/2) - i\frac{\omega-\omega_0}{\omega'}\sin(\omega' t/2))e^{i\omega t/2} & -i\frac{\tilde{\Omega}}{\omega'}\sin(\omega' t/2)e^{i\omega t/2} \\ -i\frac{\tilde{\Omega}^*\sin(\omega' t/2)}{\omega'}e^{-i\omega t/2} & (\cos(\omega' t/2) + i\frac{(\omega-\omega_0)}{\omega'}\sin(\omega' t/2))e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

Recall that $\langle \boldsymbol{\sigma} \rangle = \langle \sigma_x \rangle \hat{\mathbf{x}} + \langle \sigma_x \rangle \hat{\mathbf{y}} + \langle \sigma_z \rangle \hat{\mathbf{z}}$

$$\langle \sigma_x \rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^* b + b^* a \tag{74}$$

If at $t = 0, \sigma_x = 1$, then $a_0 = b_0 = \frac{1}{\sqrt{2}}$ then

$$a^* = \frac{1}{\sqrt{2}} \left( (\cos(\omega' t/2) + i\frac{\omega - \omega_0}{\omega'}\sin(\omega' t/2))e^{-i\omega t/2} + i\frac{\tilde{\Omega}^*}{\omega'}\sin(\omega' t/2)e^{-i\omega t/2} \right)$$

$$b = \frac{1}{\sqrt{2}} \left( -i\frac{\tilde{\Omega}^*\sin(\omega' t/2)}{\omega'}e^{-i\omega t/2} + (\cos(\omega' t/2) + i\frac{(\omega - \omega_0)}{\omega'}\sin(\omega' t/2))e^{-i\omega t/2} \right)$$

$$a^* b = \frac{1}{2} \Bigg( (\cos^2(\omega' t/2) - \frac{(\omega_0 - \omega)^2}{\omega'^2}\sin^2(\omega' t/2))e^{-i\omega t} + \frac{(\tilde{\Omega}^*)^2}{\omega'^2}\sin^2(\omega' t/2)e^{-i\omega t} + 2i\frac{\omega - \omega_0}{\omega'}\cos(\omega' t/2)\sin(\omega' t/$$

The polarization, ($\tilde{\Omega} = \Omega + i\Pi$),

$$\langle s_x \rangle = \left( \cos^2(\omega' t/2) - \frac{(\omega_0 - \omega)^2 - \Omega^2 + \Pi^2}{\omega'^2}\sin^2(\omega' t/2) \right) \cos\omega t + 2 \Bigg( -\frac{\Omega\Pi}{\omega'^2}\sin^2(\omega' t/2) - \frac{\omega_0 - \omega}{\omega'}\cos(\omega' t/2)$$

$$\langle s_y \rangle = -\left( \cos^2(\omega' t/2) - \frac{(\omega_0 - \omega)^2 - \Omega^2 + \Pi^2}{\omega'^2}\sin^2(\omega' t/2) \right) \sin\omega t + 2 \Bigg( -\frac{\Omega\Pi}{\omega'^2}\sin^2(\omega' t/2) - \frac{\omega_0 - \omega}{\omega'}\cos(\omega' t/$$

$$\langle s_z \rangle = \frac{1}{2}(|a|^2 - |b|^2) = -\frac{(\omega_0 - \omega)\Omega}{\omega'^2}\sin^2(\omega' t/2) + \frac{\Pi}{\omega'}\cos(\omega' t/2)\sin(\omega' t/2)$$

### C. Electrostatic vertical focusing in a uniform magnetic field

The equations of motion are

$$\frac{d\beta_z}{dt} = \frac{e}{\gamma mc}E_z(1 - \beta_z^2) \tag{76}$$

$$\frac{d\beta_x}{dt} = \frac{e}{\gamma mc}(\beta_y B_z - \beta_x \beta_z E_z) \tag{77}$$

$$\frac{d\beta_y}{dt} = -\frac{e}{\gamma mc}(\beta_x B_z + \beta_y \beta_z E_z) \tag{78}$$

The magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$, and the electric field $\mathbf{E} = E_z\hat{\mathbf{z}} = -kz\hat{\mathbf{z}}$. Let's substitute the proposed solution (to second order in the small parameter $(\beta_z^0)$) into the above equations of motion to confirm.

$$\boldsymbol{\beta} = -\beta_{xy}^0 \cos(\beta_z^0 \sin\omega_v t)(\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}}) - \sin(\beta_z^0 \sin\omega_v t)\hat{\mathbf{z}} \tag{79}$$

Dropping terms beyond second order in $\beta_z^0$ the first of the three equatons of motion becomes

$$\begin{aligned}\frac{d\beta_z}{dt} &= \frac{e}{\gamma mc}E_z(1-\beta_z^2) = -\frac{ek}{\gamma_0 mc}z(1-\beta_z^2)^{3/2}\\ &= -\frac{\omega_v^2}{c}z_0\cos\omega_v t(1-\beta_z^2)^{3/2} = -\omega_v\beta_z^0\cos\omega_v t(1-\beta_z^2)^{3/2} \sim -\omega_v\beta_z^0\cos\omega_v t\end{aligned}$$

The time derivative of $\beta_z$ from 79 is

$$\frac{d\beta_z}{dt} = -\beta_z^0\omega_v\cos\omega_v t\cos(\beta_z^0\sin\omega_v t) \sim -\beta_z^0\omega_v\cos\omega_v t$$

in agreeement with the preceding. The second of the three equations of motion is

$$\frac{d\beta_x}{dt} = \frac{e}{\gamma mc}(\beta_y B_z - \beta_x\beta_z E_z) = \omega\beta_y - \beta_x\beta_z\omega_v\beta_z^0$$

The time derivative of $\beta_x$ given in 79 is

$$\begin{aligned}&= \beta_{xy}^0\left(\omega\cos(\beta_z^0\sin\omega_v t)\sin\omega t + \beta_z^0\omega_v\cos\omega_v t\sin(\beta_z^0\sin\omega_v t)\cos\omega t\right)\\ &\sim -\omega\beta_y - E_z\beta_z\beta_x)\end{aligned}$$

as required.

### D. Lorentz Boost

Consider again the muon circulating in a uniform magnetic field $\mathbf{B} = B_z\hat{\mathbf{z}}$. The spiral trajectory of the muon, traveling in a clockwise direction has velocity components parallel and perpendicular to the magnetic field $\boldsymbol{\beta} = \boldsymbol{\beta}_{xy}(\cos\omega t)\ \hat{\mathbf{x}} - \sin(\omega t)\ \hat{\mathbf{y}}) + \beta_z\hat{\mathbf{z}}$ (see 3). We translate by a Lorentz boost $\beta_z$ in the z-direction into the frame where $\beta_z = 0$. The momentum transverse to the boost, and the magnetic field (parallel) to the boost are invariant. In the boosted frame the velocity is

$$\boldsymbol{\beta} = |\boldsymbol{\beta}|(\cos\omega_c^b t\hat{\mathbf{x}} - \sin\omega_c^b t\hat{\mathbf{y}}) \tag{80}$$

with cyclotron frequency is

$$\omega_c^b = \frac{eB_z}{mc\gamma'}.$$

According to the BMT equation (Equation 8), the polarization of the muon in the boosted frame will precess about the $z'$-axis with frequency

$$\omega_s^b = \frac{eB_z}{mc}(a_\mu + \frac{1}{\gamma'})$$

such that

$$\mathbf{s} = |\mathbf{s}|(\cos\omega_s^b t\hat{\mathbf{x}} - \sin\omega_s^b t\hat{\mathbf{y}})$$

where the superscript $b$ indicates the quantity measured by an observer in the boosted frame.

Then, the quantity of interest,

$$\boldsymbol{\beta}\cdot\mathbf{s} = -|\boldsymbol{\beta}||\mathbf{s}|(\cos\omega_c^b t\cos\omega_s^b t + \sin\omega_c^b t\sin\omega_s^b t) = -|\boldsymbol{\beta}||\mathbf{s}|\cos(\omega_c^b - \omega_s^b)t$$

and $\omega_a^b = \omega_s^b - \omega_c^b$.

The observer in the boosted frame measures the frequencies $\omega_a^b, \omega_s^b, \omega_c^b$ with her local clock. Now let's transform back to the lab frame. From the point of view of the observer in the lab frame, the time measured by the clock in the boosted frame is dilated. According to the clock in the lab frame everything is running slower in the boosted frame, including the frequencies of revolution and precession by a factor $1/\gamma$ where $\gamma = 1/\sqrt{1-\beta_z^2}$. As measured in the lab frame

$$\begin{aligned}\omega_a(\text{lab}) &= \omega_a^b\sqrt{1-\beta_z^2} \\ &= \omega_a^b(1-\psi^2\beta_{xy}^2)^{1/2} \\ &\sim \omega_a^b(1-\frac{1}{2}\psi^2)\end{aligned} \tag{81}$$

where $\psi$ is the pitch angle and in the limit where $\psi \ll 1$ and $\gamma \gg 1$.

### E. Magic Momentum in Electrostatic focusing channel

Consider the projection of the polarization onto the direction of motion.

$$\begin{aligned}\mathbf{s}\cdot\hat{\boldsymbol{\beta}} &= \frac{\beta_x^0\cos(\kappa\gamma_0\beta_z^0\beta_x^0\sin\omega_v t)\cos(\beta_z^0\sin\omega_v t) + \sin(\kappa\gamma_0\beta_z^0\beta_x^0\sin\omega_v t)\sin(\beta_z^0\sin\omega_v t)}{[\beta_0^{0\,2}\cos^2(\beta_z^0\sin\omega_v t) + \sin^2(\beta_z^0\sin\omega_v t)]^{1/2}} \\ &= \frac{\beta_x^0\cos\phi\cos\theta + \sin\phi\sin\theta}{[\beta_x^{0\,2}\cos^2\theta + \sin^2\theta]^{1/2}}\end{aligned}$$

Expand to second order in $\theta, \phi$

$$
\begin{aligned}
\mathbf{s}\cdot\hat{\boldsymbol{\beta}} &\approx \frac{\beta_x^0(1-\frac{1}{2}\phi^2)(1-\frac{1}{2}\theta^2)+\phi\theta}{[\beta_x^{0\,2}(1-\frac{1}{2}\theta^2)^2+\theta^2]^{1/2}} \\
&\approx \frac{\beta_x^0(1-\frac{1}{2}\phi^2)(1-\frac{1}{2}\theta^2)+\phi\theta}{[\beta_x^{0\,2}(1-\theta^2)+\theta^2]^{1/2}} \\
&\approx \frac{\beta_x^0(1-\frac{1}{2}\phi^2)(1-\frac{1}{2}\theta^2)+\phi\theta}{[\beta_x^{0\,2}+\theta^2(1-\beta_x^{0\,2})]^{1/2}} \\
&\approx \frac{1}{\beta_x^0}\left(\beta_x^0(1-\frac{1}{2}\phi^2)(1-\frac{1}{2}\theta^2)+\phi\theta\right)\left(1-\frac{\theta^2}{2\gamma_0^2\beta_x^{0\,2}}\right) \\
&\approx \frac{1}{\beta_x^0}\left(\beta_x^0(1-\frac{1}{2}(\phi^2+\theta^2))+\phi\theta\right)\left(1-\frac{\theta^2}{2\gamma_0^2\beta_x^{0\,2}}\right) \\
&\approx \frac{1}{\beta_x^0}\left(\beta_x^0(1-\frac{1}{2}(\phi^2+\theta^2))+\phi\theta-\frac{\beta_x^0\theta^2}{2\gamma_0^2\beta_x^{0\,2}}\right)
\end{aligned}
$$

Then

$$
\frac{d}{dt}\mathbf{s}\cdot\hat{\boldsymbol{\beta}} = 0
$$

if

$$
\begin{aligned}
0 &= -\frac{\beta_x^0}{2}(\phi^2+\theta^2)+\phi\theta-\frac{\beta_x^0\theta^2}{2\gamma_0^2\beta_x^{0\,2}} \\
0 &= -\frac{\beta_x^0}{2}((\kappa\gamma_0\beta_x^0)^2+1)+\kappa\gamma_0\beta_x^0-\frac{\beta_x^0}{2\gamma_0^2\beta_x^{0\,2}} \\
0 &= -\frac{1}{2}((\kappa\gamma_0\beta_x^0)^2+1)+\kappa\gamma_0-\frac{1}{2\gamma_0^2\beta_x^{0\,2}} \\
0 &= ((\kappa\gamma_0\beta_x^0)^2+1)-2\kappa\gamma_0+\frac{1}{\gamma_0^2\beta_x^{0\,2}} \\
\kappa &= \frac{\left(2\gamma_0\pm[4\gamma_0^2-4(\beta_x^{0\,2}\gamma_0^2+1)]^{1/2}\right)}{2\gamma_0^2\beta_x^{0\,2}} \\
\kappa &= \frac{1}{\gamma_0\beta_x^{0\,2}}
\end{aligned}
$$

Finally

$$
\begin{aligned}
\kappa = a_\mu+\frac{1}{\gamma_0+1} &= \frac{1}{\gamma_0\beta_x^{0\,2}} \\
\rightarrow a_\mu &= \frac{1}{\gamma_0}\frac{\gamma_0^2}{\gamma_0^2-1}-\frac{1}{\gamma_0+1} \\
a_\mu &= \frac{\gamma_0}{\gamma_0^2-1}-\frac{\gamma_0-1}{\gamma_0^2-1} \\
a_\mu &= \frac{1}{\gamma_0^2-1}
\end{aligned}
$$

### F. Determining the frequency shift with time dependent perturbation theory

Our goal is to solve the Schrodinger equation, defined by the time dependent Hamiltonian (Equation 25) using standard time dependent perturbation theory. But before we address that problem let's try something simpler, namely the case with fixed $\beta_z$ as in 12. We already know the exact solution and we can expand the exact frequency shift to second order in the small parameter $\Omega$. (Recall that $\Omega$ is proportional to the velocity in the direction of the magnetic field $\beta_z$.) We can alternatively solve the Schrodinger equation for 12 to second order in $\Omega$ using perturbation theory[1][p. 302] and check for consistency. (*Following Griffiths QM, 1st edition, section 9.1.2, page 302, problem 9.4*).

$$\begin{aligned}\mathcal{H} &= -\frac{\hbar}{2}\begin{pmatrix}\omega_0 & \Omega e^{i\omega t}\\ \Omega e^{-i\omega t} & -\omega_0\end{pmatrix}\\ &= \mathcal{H}_0 + \mathcal{H}'\\ &= -\frac{\hbar}{2}\begin{pmatrix}\omega_0 & 0\\ 0 & -\omega_0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & \Omega e^{i\omega t}\\ \Omega e^{-i\omega t} & 0\end{pmatrix}\end{aligned}$$

The Schrodinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi(t) = \mathcal{H}\psi(t)$$

$\psi_a$ and $\psi_b$ are eigenkets of the unperturbed Hamiltonian with eigenvalues $E_a = -\frac{\hbar}{2}\omega_0 = -E_b$

$$\mathcal{H}_0\psi_a = E_a\psi_a, \quad \mathcal{H}_0\psi_b = E_b\psi_b$$

and since $\psi_a$ and $\psi_b$ form a complete set, the general solution for the exact Hamiltonian is a linear combination of the two

$$\begin{aligned}\psi(t) &= c_a(t)\psi_a e^{-iE_z t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}\\ &= c_a(t)\psi_a e^{i\omega_0 t/2} + c_b(t)\psi_b e^{-i\omega_0 t/2}\end{aligned} \tag{82}$$

Substitution of 82 into the Schrodinger equation yields, after some manipulation

$$\begin{aligned}\dot{c}_a &= -\frac{i}{\hbar}c_b\mathcal{H}'_{ab}e^{-i\omega_0 t}\\ &= -ic_b\frac{\Omega}{2}e^{i\omega t}e^{-i\omega_0 t}\end{aligned}$$

or

$$
\begin{aligned}
i\dot{c}_a &= c_b \frac{\Omega}{2} e^{i\eta t} \\
i\dot{c}_b &= c_a \frac{\Omega}{2} e^{-i\eta t}
\end{aligned} \tag{83}
$$

where $\eta = \omega - \omega_0$. If the muon is initially (at $t = 0$) in an eigenstate of $\sigma_x$ the zeroth order solution to the Schrodinger equation is,

$$
\begin{aligned}
c_a(0) &= \frac{1}{\sqrt{2}} \\
c_b(0) &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

In order to compute the first order correction we substitute the zeroth order solution into the right hand side of 83 and integrate

$$
\begin{aligned}
c_a^1(t) &= \frac{-i}{\sqrt{2}} \int_0^t V e^{i\eta t} = \frac{-i}{\sqrt{2}} \frac{V}{i\eta}(e^{i\eta t} - 1) = -\frac{1}{\sqrt{2}} \frac{V}{\eta}(e^{i\eta t} - 1) \\
c_b^1(t) &= \frac{-i}{\sqrt{2}} \int_0^t V e^{-i\eta t} = \frac{-i}{\sqrt{2}} \frac{V}{-i\eta}(e^{-i\eta t} - 1) = \frac{1}{\sqrt{2}} \frac{V}{\eta}(e^{-i\eta t} - 1)
\end{aligned}
$$

where $V = \frac{\Omega}{2}$. Now we can write $c_a(t), c_b(t)$ to first order

$$
\begin{aligned}
c_a(t) &\sim c_a^0(t) + c_a^1(t) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{V}{\eta}(e^{i\eta t} - 1) \\
c_b(t) &\sim c_b^0(t) + c_b^1(t) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{V}{\eta}(e^{-i\eta t} - 1)
\end{aligned}
$$

To compute the second order term substitue $c_a^1(t)$, $c_b^1(t)$ into 83 and integrate again

$$
\begin{aligned}
i\dot{c}_a^2 &= c_b^1 V e^{-i\eta t} = \frac{1}{\sqrt{2}} \frac{V}{\eta}(e^{-i\eta t} - 1) V e^{i\eta t} \\
c_a^2 &= -\frac{i}{\sqrt{2}} \frac{V}{\eta} \int_0^t (e^{-i\eta t'} - 1) V e^{i\eta t'} dt' \\
c_a^2 &= -\frac{i}{\sqrt{2}} \frac{V}{\eta} \int_0^t (1 - e^{i\eta t'}) V dt' \\
c_a^2 &= -\frac{i}{\sqrt{2}} \frac{V}{\eta} (t - \frac{e^{i\eta t} - 1}{i\eta}) V \\
&= \frac{1}{\sqrt{2}} \frac{V^2}{\eta} \left( \frac{e^{i\eta t} - 1}{\eta} - it \right) \\
c_b^2 &= \frac{1}{\sqrt{2}} \frac{V^2}{\eta} \left( \frac{e^{-i\eta t} - 1}{\eta} + it \right)
\end{aligned} \tag{84}
$$

Following Griffith's notation where $a(t) = c_a(t) e^{-i\omega_0 t/2}$ and $b(t) = c_b(t) e^{i\omega_0 t/2}$ we can write

$$
a(t) \sim \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} \left( 1 - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2} \left( e^{i\eta t} - 1 - i\eta t \right) + \dots \right) \tag{85}
$$

Rearranging

$$a(t) \sim \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} \left( [1 - i\frac{V^2}{\eta} t] - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2}\left(e^{i\eta t} - 1\right) + \ldots \right)$$

Then to order $V^2$

$$\begin{aligned}
a(t) &\sim \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} [1 - i\frac{V^2}{\eta} t] \left( 1 - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2}\left(e^{i\eta t} - 1\right) + \ldots \right) \\
&\sim \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} e^{-i\frac{V^2}{\eta} t} \left( 1 - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2}\left(e^{i\eta t} - 1\right) + \ldots \right) \\
&\sim \frac{1}{\sqrt{2}} e^{i(\omega_0 - \omega)t/2} e^{i\omega t/2} e^{-i\frac{V^2}{\eta} t} \left( 1 - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2}\left(e^{i\eta t} - 1\right) + \ldots \right) \\
&\sim \frac{1}{\sqrt{2}} e^{-i\lambda t/2} e^{i\omega t/2} \left( 1 - \frac{V}{\eta}(e^{i\eta t} - 1) + \frac{V^2}{\eta^2}\left(e^{i\eta t} - 1\right) + \ldots \right) \qquad (86)
\end{aligned}$$

where $\lambda/2 = \frac{\eta}{2} + \frac{V^2}{\eta}$. Finally with the substitutios $\eta = \omega - \omega_0$ and $V = \Omega/2$

$$\begin{aligned}
a(t) &\sim \frac{1}{\sqrt{2}} e^{-i(\omega - \omega_0 + \frac{\Omega^2}{2(\omega-\omega_0)})t/2} e^{i\omega t/2} \left( 1 - \frac{\Omega}{4(\omega - \omega_0)}(e^{i(\omega-\omega_0)t} - 1) + \frac{\Omega^2}{4(\omega-\omega_0)^2}\left(e^{i(\omega-\omega_0)t} - 1\right) + \ldots \right) \\
&\sim \frac{1}{\sqrt{2}} e^{-i(\omega - \omega_0 + \frac{\Omega^2}{2(\omega-\omega_0)})t/2} e^{i\omega t/2} \left( 1 - \left( \frac{\Omega}{(\omega-\omega_0)} - \frac{\Omega^2}{2(\omega-\omega_0)^2} \right) i e^{i(\omega-\omega_0)t/2} \sin((\omega-\omega_0)t/2) + \ldots \right) \\
&\sim \frac{1}{\sqrt{2}} e^{-i(\frac{\Omega^2}{2(\omega-\omega_0)})t/2} e^{i\omega t/2} \left( e^{-i(\omega-\omega_0)t/2} - \left( \frac{\Omega}{(\omega-\omega_0)} - \frac{\Omega^2}{2(\omega-\omega_0)^2} \right) i \sin((\omega-\omega_0)t/2) + \ldots \right) \qquad (87) \\
&\sim \frac{1}{\sqrt{2}} e^{i\omega t/2} \left( e^{-i(\omega - \omega_0 + \frac{\Omega^2}{2(\omega-\omega_0)})t/2} - \left( \frac{\Omega}{(\omega-\omega_0)} - \frac{\Omega^2}{2(\omega-\omega_0)^2} \right) i \sin((\omega-\omega_0)t/2) + \ldots \right) \qquad (88)
\end{aligned}$$

To get from 87 to 88 we drop terms of order $> \Omega^2$.

Let's compare 88 to the exact solution 4 with initial conditions $a_0 = b_0 = \frac{1}{\sqrt{2}}$, expanded to second order in $\Omega$.

$$a = \frac{1}{\sqrt{2}} \left( \cos(\omega' t/2) - \frac{i(\omega - \omega_0)}{\omega'} \sin(\omega' t/2) - \frac{i\Omega}{\omega'} \sin(\omega' t/2) \right) e^{i\omega t/2}$$

Note $\omega' = \sqrt{(\omega-\omega_0)^2 + \Omega^2} \sim \omega - \omega_0 + \frac{\Omega^2}{2(\omega-\omega_0)}$. With that approximation

$$\begin{aligned}
a &\sim \frac{1}{\sqrt{2}} \left( \cos(\omega' t/2) - i(1 - \frac{\Omega^2}{2(\omega-\omega_0)^2}) \sin(\omega' t/2) - \frac{i\Omega}{\omega - \omega_0} \sin(\omega' t/2) \right) e^{i\omega t/2} \\
&\sim \frac{1}{\sqrt{2}} \left( e^{-i(\omega - \omega_0 + \frac{\Omega^2}{2(\omega-\omega_0)})t/2} + i(\frac{\Omega^2}{2(\omega-\omega_0)^2}) \sin((\omega-\omega_0)t/2) - \frac{i\Omega}{\omega-\omega_0} \sin((\omega-\omega_0)t/2) \right) e^{i\omega t/2} \qquad (89)
\end{aligned}$$

Sure enough the exact solution expanded to second order in $\Omega$ (Equation 89) is consistent with the perturbative solution (Equation 88). The important take away is that the second order frequency shift is the coefficient of the term linear in imaginary time ($it$). (See Equation 84 - 86)

The frequency derived from the perturbation theory

$$\lambda = \eta + 2\frac{V^2}{\eta} = (\omega_0 - \omega) + \frac{1}{2}\frac{\Omega^2}{(\omega_0 - \omega)} \tag{90}$$

and the exact frequency (Equation 14)

$$\omega' = \left((\omega_0 - \omega)^2 + \Omega^2\right)^{1/2} \sim (\omega_0 - \omega) + \frac{1}{2}\frac{\Omega^2}{(\omega_0 - \omega)^2} \tag{91}$$

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