CONVERTING BETWEEN BMAD AND MAD-X TRANSVERSE COUPLING REPRESENTATIONS

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Abstract

Bmad uses a version of the Edwards-Teng representation for transverse coupling, while MAD-X (through `PTC_TWISS`) uses a version of the Ripken representation. I give formulas for converting between the two representations.

THE NORMALIZING MATRIX

When discussing Courant-Snyder lattice functions or their generalization to the coupled case, we are really referring to is a parameterization of the normalizing matrix for the linear map of the transverse variables. Thus, the linear map from point 1 to point 2 is decomposed as

$$M_{21} = A_2 R_{21} A_1^{-1} \tag{1}$$

where all matrices are symplectic, and

$$R_{21} = \begin{bmatrix} \cos\phi_{a21} & \sin\phi_{a21} & 0 & 0 \\ -\sin\phi_{a21} & \cos\phi_{a21} & 0 & 0 \\ 0 & 0 & \cos\phi_{b21} & \sin\phi_{b21} \\ 0 & 0 & -\sin\phi_{b21} & \cos\phi_{b21} \end{bmatrix} \tag{2}$$

From this definition, $\phi_{\cdot 21}$ are arbitrary since both $A_\cdot$ can be multiplied on the right hand side by a matrix of the same form as R_{21} and the normalizing relation will still have the same form. For a periodic line, $A_1 = A_2$, therefore $\phi_{\cdot 21}$ are no longer arbitrary, but the freedom in multiplying A_1 on the right by a matrix of the same form as R_{21} still exists. A symplectic 4×4 matrix has 10 degrees of freedom, and the freedom of choice of the two rotation angles means that there are 8 degrees of freedom in $A_\cdot$.

The Ripken representation [1, 2] describes those degrees of freedom with β_{ij} and α_{ij}, $i, j \in \{1, 2\}$, defined as

$$\beta_{ij} = A_{2i-1,2j-1}^2 + A_{2i-1,2j}^2 \tag{3}$$

$$\alpha_{ij} = -A_{2i-1,2j-1}A_{2i,2j-1} - A_{2i-1,2j}A_{2i,2j} \tag{4}$$

For convenience we also define

$$\gamma_{ij} = A_{2i,2j-1}^2 + A_{2i,2j}^2 \tag{5}$$

Note that the indices of A are numbered from 1 to 4. For the uncoupled case, $\beta_{11} = \beta_x$, $\beta_{22} = \beta_y$, $\beta_{12} = \beta_{21} = 0$, $\alpha_{11} = \alpha_x$, $\alpha_{22} = \alpha_y$, and $\alpha_{12} = \alpha_{21} = 0$.

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The Edwards-Teng representation, as implemented in Bmad [3], describes A as

$$A = \begin{bmatrix} \gamma & 0 & C_{11} & C_{12} \\ 0 & \gamma & C_{21} & C_{22} \\ -C_{22} & C_{12} & \gamma & 0 \\ C_{21} & -C_{11} & 0 & \gamma \end{bmatrix} \begin{bmatrix} \beta_a^{1/2} & 0 & 0 & 0 \\ -\alpha_a\beta_a^{-1/2} & \beta_a^{-1/2} & 0 & 0 \\ 0 & 0 & \beta_b^{1/2} & 0 \\ 0 & 0 & -\alpha_b\beta_b^{-1/2} & \beta_b^{-1/2} \end{bmatrix} \tag{6}$$

where

$$\gamma^2 + C_{11}C_{22} - C_{12}C_{21} = 1 \tag{7}$$

The 8 parameters are the 4 C_{ij}, the 2 β_i and the 2 α_i.

CONVERSION BETWEEN REPRESENTATIONS

Converting from the Edwards-Teng representation to the Ripken representation is straightforward. Simply multiply out the matrices in Eq. (6) and perform the computation in Eq. (4). The results are

$$\beta_{11} = \gamma^2\beta_a \tag{8}$$

$$\alpha_{11} = \gamma^2\alpha_a \tag{9}$$

$$\beta_{22} = \gamma^2\beta_b \tag{10}$$

$$\alpha_{22} = \gamma^2\alpha_b \tag{11}$$

$$\beta_{21} = c_{22}^2\beta_a + 2c_{12}c_{22}\alpha_a + c_{12}^2\gamma_a \tag{12}$$

$$\alpha_{21} = (c_{11}c_{22} + c_{12}c_{21})\alpha_a + c_{21}c_{22}\beta_a + c_{11}c_{12}\gamma_a \tag{13}$$

$$\beta_{12} = c_{11}^2\beta_b - 2c_{11}c_{12}\alpha_b + c_{12}^2\gamma_b \tag{14}$$

$$\alpha_{12} = (c_{11}c_{22} + c_{12}c_{21})\alpha_b - c_{11}c_{21}\beta_b - c_{12}c_{22}\gamma_b \tag{15}$$

Here $\gamma_i = (1 + \alpha_i^2)/\beta_i$.

The auxiliary parameters u, ν_1, and ν_2 from the Mais-Ripken formulation in [2] can also be computed:

$$u = 1 - \gamma^2 \tag{16}$$

$$\beta_{21}^{1/2}\cos\nu_1 = -\beta_a^{1/2}c_{22} - c_{12}\alpha_a\beta_a^{-1/2} \tag{17}$$

$$\beta_{21}^{1/2}\sin\nu_1 = -\beta_a^{-1/2}c_{12} \tag{18}$$

$$\beta_{12}^{1/2}\cos\nu_2 = \beta_b^{1/2}c_{11} - \alpha_b\beta_b^{-1/2}c_{12} \tag{19}$$

$$\beta_{12}^{1/2}\sin\nu_2 = -\beta_b^{-1/2}c_{12} \tag{20}$$

Inverting the transformation is more complex. First, note that

$$\beta_{21}\gamma_{11} - \alpha_{11}^2 = \beta_{22}\gamma_{22} - \alpha_{22}^2 = \gamma^4 \tag{21}$$

$$\beta_{21}\gamma_{21} - \alpha_{21}^2 = \beta_{12}\gamma_{12} - \alpha_{12}^2 = (1 - \gamma^2)^2 \tag{22}$$

One can thus find γ and therefore β_a, α_a, β_b, and α_b.

REFERENCES

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